DIOPHANTINE EXPONENTS OF TWO-DIMENSIONAL MAHLER VECTORS

REYNOLD FREGOLI AND YOUYI ZHANG

Abstract. We study the irrationality exponent of two-dimensional vectors obtained by evaluating solutions to a Mahler system at special rational values. We establish an upper bound for the irrationality exponent of such vectors, under a non-vanishing condition on the associated sequence of Hankel determinants. Further, we construct examples of vectors whose irrationality exponent equals $\frac{3}{2}$, the minimum possible exponent. Our proofs combine type I Hermite–Padé approximation with Hankel determinants.

1. Introduction

1.1. **Mahler Numbers and Diophantine Approximation.** Let $\mathbb{Q}[[z]]$ denote the ring of formal power series in the variable z with coefficients in $\mathbb{Q}$ and let $\mathbb{Q}(z)$ denote the corresponding field of rational functions.

In its simplest form, a Mahler equation is a functional equation of the form

$$f(z) = a(z)f(z^d) + b(z), \tag{1.1}$$

where $a, b \in \mathbb{Q}(z)$ and $f \in \mathbb{Q}[[z]]$. Such equations were first introduced by Mahler [Mah29] as a tool to study power series f with lacunary terms and, specifically, to establish the irrationality of numbers of the form $f(p/q)$ $(p, q \in \mathbb{N})$. A very useful survey of Mahler's foundational work in transcendence theory and what later became known as Mahler's method can be found in [Ada19]

A surprising connection exists between Mahler equations and the theory of automatic sequences. These are sequences of non-negative integers that can be produced by abstract machines known as finite automata (see [AS03, Sections 4 and 5] and [BHWY16, page 2] for a precise definition). Real numbers whose sequence of digits in base b is an automatic sequence (a_n) are known in the literature as *automatic numbers* and may be viewed as those numbers whose digit expansion is completely deterministic.

In can be shown that the generating function $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ of an automatic sequence a_n always satisfies a Mahler equation (slightly more general than (1.1)) [Cob68, page 57][Ada19, Section 7], so that if α can be written as $a_0.a_1a_2\cdots$ in base b, then $\alpha = f(1/b)$, with f essentially as in (1.1). This observation together with a

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modern reformulation of Mahler's method ultimately led to a proof of the striking fact that all automatic numbers are either rational or transcendental [AF17, Corollaire 1.8][1].

The extensive work on the transcendence of automatic numbers and, more generally, of numbers of the form $f(p/q)$, with f as in (1.1), naturally motivates the study of their Diophantine properties. For example, the irrationality exponent of $\xi \in \mathbb{R}$ is defined as

$$\mu(\xi) := \sup\left\{\mu > 0 : \left|\xi - \frac{p}{q}\right| < q^{-\mu} \text{ for infinitely many rationals } \frac{p}{q}\right\} \geq 2.$$

For transcendental numbers, computing the irrationality exponent can be an extremely challenging task (see for example [FR25]). Nonetheless, it is known that many automatic numbers have exponent 2 and thus provide a large class of transcendental numbers whose irrationality exponent is minimal. This was established in [Bug11] for the Thue-Morse constant (a well-known automatic number) and in [BHWY16] for different Mahler numbers.

In [BZ15], Badziahin and Zorin show that the Thue-Morse constant (together with other Mahler numbers) is further well-approximable and pose two interesting open problems. The second of these is the starting point of this paper. Let us briefly recall it.

In place of (1.1), one may consider a more general functional equation in several indeterminates, known as a Mahler system. This is a system of the form

$$\underline{f}(z) = A(z)\underline{f}(z^d) + B(z), \tag{1.2}$$

where $\underline{f}(z) \in (\mathbb{Q}[[z]])^{s\times 1}$, $A(z) \in (\mathbb{Q}(z))^{s\times s}$, and $B(z) \in (\mathbb{Q}(z))^{s\times 1}$, with $s, d \geq 1$ fixed integers. Then the following is [BZ15, Open Problem 2].

Problem 1.1. Let $\underline{f}$ be an analytic solution to (1.2) and assume that its components f_i are transcendental[2] over $\mathbb{Q}(z)$. What is the irrationality exponent of the numbers $f_i(r)$ for $i = 1, \dots, k$ and $r \in \mathbb{Q}$? Are these numbers badly approximable?

In this paper, we revisit this problem from a different angle, studying instead Diophantine properties of $\underline{f}(r)$ as a vector. Namely, we ask:

Problem 1.2. Let $\underline{f}(z) \in (\mathbb{Q}[[z]])^s$ be as in Problem 1.1. What are the Diophantine properties of the vector $\underline{f}(r)$, where $r \in \mathbb{Q}$? What are its simultaneous and dual Diophantine exponents (see below for the definition)? Is this vector badly approximable?

1.2. **Main Results.** In order to make our main results precise, we require a few definitions. Let $s \geq 1$ be an integer. The simultaneous Diophantine exponent of a vector $\underline{\xi} \in \mathbb{R}^s$ is defined as

$$\omega(\underline{\xi}) = \sup\left\{\omega > 0 : \|q\underline{\xi} - \underline{p}\| < q^{-\omega} \text{ for infinitely many } (q, \underline{p}) \in \mathbb{N} \times \mathbb{Z}^s\right\},$$

where $\|\cdot\|$ denotes the supremum norm. The irrationality exponent of $\underline{\xi}$ is then $\mu(\underline{\xi}) = \omega(\underline{\xi}) + 1$. For $(A_0, A_1, \dots, A_s) \in \mathbb{Z}^{s+1}$, put

[1] It should be remarked that the first proof of this fact appears in [AB07] and, in fact, the result is also claimed by Cobham [Cob68, page 57], without proof.

[2] By this we mean that no f_i satisfies a polynomial equation whose coefficients are in $\mathbb{Q}(z)$.

$$H(A_0,\dots,A_s)=\max\{|A_0|,\dots,|A_s|\}.$$

The dual Diophantine exponent of $\underline{\xi}\in\mathbb{R}^s$ is defined as

$$\omega^*(\underline{\xi})\sup\Big\{\omega>0:|A_0+A_1\xi_1+\cdots+A_s\xi_s|<H(A_0,\dots,A_s)^{-\omega} \\ \text{for infinitely many } (A_0,\dots,A_s)\in\mathbb{Z}^{s+1}\setminus\{\underline{0}\}\Big\}.$$

Remark 1.3. It is well-known that for all $\underline{\xi}\in\mathbb{R}^s$ one has $\omega(\underline{\xi})\geq 1/s$ and $\omega^*(\underline{\xi})\geq s$. In particular, $\mu(\underline{\xi})\geq 1+1/s$ for all $\underline{\xi}\in\mathbb{R}^s$.

In order to state our first theorem, we further need to introduce the notion of Hankel determinant. This is a classical tool in the study of continued fractions for power series (see [BHWY16]). For the sake of compactness, we introduce this definition and state our main result in the case $s=2$, although our techniques may be used to extend the statement to the general case $s\geq 2$ with relatively little effort.

Definition 1.4 (Type I Hankel determinants)**.** Let $\underline{f}=(f_1,f_2)\in(\mathbb{Q}[[z]])^2$. Write

$$f_1(z)=\sum_{k=0}^{\infty}c_kz^k\quad\text{and}\quad f_2(z)=\sum_{k=0}^{\infty}d_kz^k. \tag{1.3}$$

The Type I Hankel determinant of $\underline{f}$ of order k is the determinant

$$H_k(f_1,f_2)=\begin{vmatrix} c_0 & c_1 & \cdots & c_k & d_0 & d_1 & \cdots & d_k\\ c_1 & c_2 & \cdots & c_{k+1} & d_1 & d_2 & \cdots & d_{k+1}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ c_{2k} & c_{2k+1} & \cdots & c_{3k} & d_{2k} & d_{2k+1} & \cdots & d_{3k}\\ c_{2k+1} & c_{2k+2} & \cdots & c_{3k+1} & d_{2k+1} & d_{2k+2} & \cdots & d_{3k+1}\end{vmatrix}.$$

Our first main result is an extension of [BHWY16, Theorem 4.1], providing upper bounds for the Diophantine exponents of vectors obtained by evaluating solutions $\underline{f}$ to (1.2) at reciprocals of integers.

Theorem 1.5. *Let $d\geq 2$ and $b\geq 2$ be integers. Let*

$$\underline{f}(z)=(f_1(z),f_2(z))\in\mathbb{Z}[[z]]^2$$

such that both f_1 and f_2 converge absolutely inside the unit disk, and suppose that $\underline{f}$ satisfies the Mahler system

$$\underline{f}(z)=\frac{C(z)}{D(z)}\underline{f}(z^d)+\frac{A(z)}{B(z)},$$

where $B(z),D(z)\in\mathbb{Q}[z]$, $C(z)\in(\mathbb{Q}[z])^{2\times 2}$, and $A(z)\in(\mathbb{Q}[z])^{2\times 1}$. Assume that

$$B\left(\frac{1}{b^{d^m}}\right)D\left(\frac{1}{b^{d^m}}\right)\det C\left(\frac{1}{b^{d^m}}\right)\neq 0\quad\textit{for all } m\geq 0$$

and that there exists a strictly increasing sequence $n_1 < n_2 < \cdots$ *such that* $H_{n_i}(f_1, f_2) \neq 0$ *for all* $i \geq 1$, *with*

$$\limsup_{i\to\infty} \frac{n_{i+1}}{n_i} = \rho < \infty. \tag{1.4}$$

Then $1, f_1(1/b), f_2(1/b)$ *are linearly independent over* $\mathbb{Q}$, *and*

$$\omega^*(\underline{f}(1/b)) \leq 2\rho^2.$$

Note that if $\rho = 1$ in Theorem 1.5, then $\omega^*(\underline{f}(1/b)) = 2$ for any $b \geq 2$. Furthermore, according to Khintchine's transference inequalities [Khi26, Satz VI, Eq. (32)], one has that

$$\frac{\omega^*(\underline{\xi})}{(s-1)\omega^*(\underline{\xi}) + s} \leq \omega(\underline{\xi}) \leq \frac{\omega^*(\underline{\xi}) - s + 1}{s}, \tag{1.5}$$

if $1, \xi_1, \ldots, \xi_s$ are linearly independent over $\mathbb{Q}$. Thus, Theorem 1.5 yields the following corollary.

Corollary 1.6. *Let* $\underline{f}$ *be as in Theorem 1.5 and assume that* $\rho = 1$. *Then for any* $b \geq 2$

$$\omega(\underline{f}(1/b)) = \frac{1}{2} \quad \textit{and} \quad \mu(\underline{f}(1/b)) = \frac{3}{2}.$$

Remark 1.7. It can be shown that any vector of the form $\underline{f}(1/b)$, where $\underline{f}$ satisfies the hypotheses of Theorem 1.5, must have transcendental components that are linearly independent over $\mathbb{Q}$. Independence follows from the fact that the Hankel determinants are non-vanishing (see Lemma 2.1) combined with the lifting theorem [AF17, Theorem 1.4] (for more details see the argument at the beginning of Section 3). To see that $f_1(1/b)$ and $f_2(1/b)$ are transcendental, first observe that the vector $(1, f_1(z), f_2(z))^T$ trivially satisfies a linear Mahler system of the form

$$\begin{pmatrix} 1 \\ \underline{f}(z) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ A(z)/B(z) & C(z)/D(z) \end{pmatrix} \begin{pmatrix} 1 \\ \underline{f}(z^d) \end{pmatrix}.$$

Then, by [Nis96, Theorem 5.1.7], both $f_1(z)$ and $f_2(z)$ are either rational or transcendental and, since they are linearly independent over $\mathbb{Q}(z)$, they must be transcendental. The conclusion then follows by [Nis90, Corollary 2]. Note that the stronger fact that the functions $1, f_1(z), f_2(z)$ are algebraically independent does not follow.

In view of Corollary 1.6, it becomes particularly interesting to construct Mahler systems such that condition (1.4) holds with $\rho = 1$. We are able to construct such examples by carefully choosing the matrix C.

A classical example of a multidimensional Mahler system is the following:

$$\underline{f}(z) = \begin{pmatrix} 1 & -z \\ 1 & z \end{pmatrix} \underline{f}(z^2).$$

However, even for this simple system, it is unclear whether Condition (1.4) with $\rho = 1$ holds. To circumvent this obstruction, we consider a slight modification of this system, of the form

$$\underline{f}(z) = \begin{pmatrix} 1 & \alpha z \\ 1 & \beta z \end{pmatrix} \underline{f}(z^2), \tag{1.6}$$

with α, β real numbers. Our second main result gives conditions on α and β under which $\rho = 1$.

Theorem 1.8. *Let p be a prime number and let $\alpha, \beta \in \mathbb{Z}$ be such that $p \nmid \alpha$ and $p \mid \beta$. Let $\underline{f} \in \mathbb{Z}[[z]]^2$ be a nonzero solution to the linear Mahler system in* (1.6) *with this choice of α and β. Then Condition* (1.4) *holds with $\rho = 1$. In particular, for any integer $b \geq 2$ we have*

$$\mu(\underline{f}(1/b)) = \frac{3}{2}.$$

Thus, we are able to construct countably many distinct automatic two-dimensional vectors whose irrationality exponent is minimal.

1.3. **AI Disclosure.** Artificial Intelligence was used to prove Lemmas 2.2 and 4.1 in this manuscript. The authors take full responsibility for the correctness of the arguments and the quality of the mathematical exposition.

2. Preliminary Results

We begin with a simple observation.

Lemma 2.1. *Let $\underline{f} \in \mathbb{Z}[[z]]^2$ and assume that the sequence $H_k(\underline{f})$ is not eventually null. Then $1, f_1(z), f_2(z)$ are linearly independent over $\mathbb{C}(z)$.*

Proof. Let f_1, f_2 be as in (1.3) and suppose that $1, f_1(z), f_2(z)$ are not linearly independent over $\mathbb{C}(z)$. Then there exists $P_0, P_1, P_2 \in \mathbb{C}(z)$ not all zero such that

$$P_0(z) + P_1(z)f_1(z) + P_2(z)f_2(z) = 0.$$

Denote

$$P_0(z) = \sum_{i=0}^{N_0} \alpha_i z^i \qquad P_1(z) = \sum_{i=0}^{N_1} \beta_i z^i \qquad P_2(z) = \sum_{i=0}^{N_2} \gamma_i z^i.$$

Then for all $k > \max_{j=0,1,2} N_j$,

$$\beta_{N_1} c_{k-N_1} + \cdots + \beta_0 c_k + \gamma_{N_2} d_{k-N_2} + \cdots + \gamma_0 d_k = 0$$

and

$$H_k(f_1, f_2) = \begin{vmatrix} c_0 & \cdots & c_{k-N_1} & \cdots & c_k & d_0 & \cdots & d_{k-N_2} & \cdots & d_k \\ c_1 & \cdots & c_{k+1-N_1} & \cdots & c_{k+1} & d_1 & \cdots & d_{k+1-N_2} & \cdots & d_{k+1} \\ \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ c_{2k+1} & \cdots & c_{3k+1-N_1} & \cdots & c_{3k+1} & d_{2k+1} & \cdots & d_{3k+1-N_2} & \cdots & d_{3k+1} \end{vmatrix} = 0.$$

This contradicts the assumption that the sequence $H_k(\underline{f})$ is not eventually null. Hence, $1, f_1(z), f_2(z)$ are linearly independent over $\mathbb{C}(z)$. □

We now prove one of our key lemmas, which is a generalization of [AR09, Lemma 4.1] to higher dimension.

Lemma 2.2. *Let $\underline{\xi} = (\xi_1, \cdots, \xi_s)$ such that $1, \xi_1, \cdots, \xi_s$ are linearly independent over $\mathbb{Q}$. Assume that for all sufficiently large n there are $s+1$ integer vectors $\underline{p}_{n,j} \in \mathbb{Z}^{s+1}$ for $j = 0, \ldots, s$ that satisfy the following:*

(1) $\det \begin{pmatrix} \underline{p}_{n,0} \\ \underline{p}_{n,1} \\ \vdots \\ \underline{p}_{n,s} \end{pmatrix} \neq 0$, *where* $\underline{p}_{n,j} = (p_{n,j,0}, p_{n,j,1}, \cdots, p_{n,j,s})$.

(2) *There exists an increasing sequence $H_n \geq \max_{0\leq j\leq s} \|\underline{p}_{n,j}\|_\infty$ with $\lim_{n\to\infty} H_n = \infty$, and there exist constants $c_0, c_1 > 0$, $\theta \geq 1$, and $\delta > s-1$ such that*

a) $H_n < H_{n+1} \leq c_0 H_n^\theta$.

b) $\max_{0\leq j\leq s} |p_{n,j,0} + p_{n,j,1}\xi_1 + \ldots, +p_{n,j,s}\xi_s| \leq c_1 H_n^{-\delta}$.

Then

$$\omega^*(\underline{\xi}) \leq \frac{s\theta}{\delta - s + 1}.$$

Proof. Take an arbitrary integer vector $\underline{p} = (p_0, p_1, \cdots, p_s) \neq \underline{0}$ and put

$$H = \|\underline{p}\|_\infty = \max\{|p_0|, |p_1|, \cdots, |p_s|\}.$$

Let $\hat{\underline{\xi}} = (1, \xi_1, \ldots, \xi_s)$ and consider the linear forms

$$L_{n,j}(\underline{x}) = p_{n,j,0}x_0 + \cdots + p_{n,j,s}x_s, \text{ for } j = 0, \ldots, s, \quad L(\underline{x}) = p_0x_0 + \cdots + p_nx_n \tag{2.1}$$

for $\underline{x} \in \mathbb{R}^{s+1}$. Note that $L(\hat{\underline{\xi}}) \neq 0$ because $1, \xi_1, \cdots, \xi_s$ are linearly independent over $\mathbb{Q}$. Now, recall that for all large enough n we have that $\det \begin{pmatrix} \underline{p}_{n,0} \\ \underline{p}_{n,1} \\ \vdots \\ \underline{p}_{n,s} \end{pmatrix} \neq 0$. Then at least one of the $s+1$ determinants where $\underline{p}$ replaces the j-th row

$$\Delta_{n,j} = \det \begin{pmatrix} \underline{p}_{n,0} \\ \vdots \\ \underline{p} \\ \vdots \\ \underline{p}_{n,s} \end{pmatrix} \quad \text{for } j = 0, \ldots, s,$$

is nonzero. For such a j, since all entries are integers, we have that

$$|\Delta_{n,j}| \geq 1.$$

By replacing the first column with a linear combination of the last s columns, we see that

$$\Delta_{n,j} = \begin{vmatrix} L_{n,0}(\hat{\underline{\xi}}) & p_{n,0,1} & \cdots & p_{n,0,s} \\ \vdots & \vdots & & \vdots \\ L(\hat{\underline{\xi}}) & p_1 & \cdots & p_s \\ \vdots & \vdots & & \vdots \\ L_{n,s}(\hat{\underline{\xi}}) & p_{n,s,1} & \cdots & p_{n,s,s} \end{vmatrix}.$$

Then, expanding the determinant along the first column, we deduce that

$$|\Delta_{n,j}| \le c_2 \left(|L(\hat{\underline{\xi}})| H_n^s + H H_n^{s-1} \max_{0 \le r \le s} |L_{n,r}(\hat{\underline{\xi}})| \right),$$

where $c_2 > 0$ depends only on s. Finally, using

$$\max_{0 \le r \le s} |L_{r,n}(\hat{\underline{\xi}})| \le c_1 H_n^{-\delta},$$

we conclude that for all sufficiently large n

$$1 \le c_2 |L(\hat{\underline{\xi}})| H_n^s + c_2 c_1 H H_n^{s-1-\delta}. \tag{2.2}$$

Now, choose $K > 0$ large enough so that $c_2 c_1 K^{-1} \le 1/2$. Since H_n is strictly increasing with $\lim_{n\to\infty} H_n = \infty$, there exists some n such that

$$H_{n-1}^{\delta-s+1} < KH \le H_n^{\delta-s+1}. \tag{2.3}$$

We will from now on assume that H is so large that when (2.3) holds, also (2.2) holds. This will not affect the conclusion, since we may exclude finitely many vectors $\underline{p}$.

From the upper bound in (2.3) and the fact that $c_2 c_2 K^{-1} \le 1/2$ we obtain

$$c_2 c_1 H H_n^{s-1-\delta} \le \frac{1}{2}.$$

Hence, by (2.2),

$$1 \le c_2 |L(\hat{\underline{\xi}})| H_n^s + \frac{1}{2}$$

and so

$$|L(\hat{\underline{\xi}})| \ge \frac{1}{2c_2} H_n^{-s}. \tag{2.4}$$

From the lower bound in (2.3), we see that

$$H_{n-1} < (KH)^{1/(\delta-s+1)}.$$

Combining this with the growth condition $H_n \le c_0 H_{n-1}^{\theta}$, we obtain

$$H_n \le c_0 (KH)^{\theta/(\delta-s+1)},$$

whence

$$H_n^{-s} \ge c_0^{-s} K^{-s\theta/(\delta-s+1)} H^{-s\theta/(\delta-s+1)}.$$

Consequently, there exists a constant $c > 0$, independent of $\underline{p}$, such that

$$|L(\hat{\underline{\xi}})| \ge c H^{-s\theta/(\delta-s+1)}.$$

Thus, for any nonzero integer vector $\underline{p} \in \mathbb{Z}^{s+1}$ with sufficiently large $H = \max\{|p_0|, \dots, |p_s|\}$, we have that

$$|p_0 + p_1\xi_1 + \cdots + p_s\xi_s| \geq cH^{-s\theta/(\delta-s+1)}.$$

By the definition of dual Diophantine exponent, this implies that

$$\omega^*(\underline{\xi}) \leq \frac{s\theta}{\delta - s + 1}.$$

□

Remark 2.3. Note that when $s = 2$, the upper bound for $\omega^*(\underline{\xi})$ is $2\theta/(\delta - 1)$.

Remark 2.4. The above lemma is based on ideas from Nesterenko's linear independence criterion [Nes85]. In fact, the case $\theta = 1$ follows directly from the reformulation of Nesterenko's criterion in [Fis15, Proposition 1]. Let us briefly explain how to see this.

Make the same assumptions as in Lemma 2.2 and let $\theta = 1$. Put $\underline{e} = (1, \underline{\xi})$. Then [Fis15, Proposition 1 Part b)], applied with $k = 1$, $p = s$, and $\tau_1 = \delta$, gives: for any $\varepsilon > 0$ and any Q sufficiently large (in terms of ε), there are non non-zero integer vectors $\underline{v}$ that can be written in the form $\underline{v} = \lambda\underline{e} + \underline{u}$ with $\|\underline{u}\|_2 < Q^{-1-\varepsilon}$ and $|\lambda| \leq Q^{(\delta-\varepsilon)}$. Note that we are replacing Q_n in [Fis15, Proposition 1 Part b)] with arbitrary Q. This modification follows from the fact that $\theta = 1$, as explained in the comment immediately following Proposition 1 in [Fis15].

We will show that $\omega(\underline{\xi}) \leq \delta^{-1}$ by contradiction. Thanks to (1.5), this will imply that $\omega^*(\underline{\xi}) \leq s/(\delta - s + 1)$ when $\delta > s - 1$.

Suppose by contradiction that

$$\omega(\underline{\xi}) > \frac{1}{\delta}.$$

Choose $\omega(\underline{\xi}) > \omega_0 > \delta^{-1}$. Then there are infinitely many integers $q \geq 1$ and $p_1, \dots, p_s \in \mathbb{Z}$ such that

$$\left\|q\underline{\xi} - \underline{p}\right\|_\infty < q^{-\omega_0}.$$

Write $\underline{v} = (q, p_1, \dots, p_s) = q\underline{e} + \underline{r}$, where $\underline{r} = (0, p_1 - \xi_1 q, \dots, p_s - \xi_s q)$. Then

$$\|\underline{r}\|_2 < \sqrt{s}q^{-\omega_0}.$$

One can also write $\underline{v} = \lambda\underline{e} + \underline{u}$ for some $\lambda \in \mathbb{R}$ and $\underline{u} \in \mathrm{Span}(\underline{e})^\perp$. Then

$$\|\underline{u}\|_2 \leq \|\underline{r}\|_2 < \sqrt{s}q^{-\omega_0},$$

since $\underline{u}$ is the orthogonal projection of $\underline{r}$ onto $\mathrm{Span}(\underline{e})^\perp$. Also,

$$\lambda = \frac{\underline{v} \cdot \underline{e}}{\|\underline{e}\|_2^2} = q + \frac{\underline{r} \cdot \underline{e}}{\|\underline{e}\|_2^2}.$$

So that

$$|\lambda - q| \leq \frac{\|\underline{r}\|_2}{\|\underline{e}\|_2} \leq q.$$

for sufficiently large q. Then $|\lambda| \leq 2q$. Take $\varepsilon > 0$ sufficiently small such that $\delta - 2\varepsilon > 0$ and $\omega_0(\delta - 3\varepsilon) > 1 + \varepsilon$. Let

$$Q = q^{\frac{1}{\delta-2\varepsilon}}.$$

Then $q = Q^{(\delta-2\varepsilon)}$. This implies that, for sufficiently large q in terms of ε, and thus sufficiently large Q, we have

$$\|\underline{u}\|_2 < \sqrt{s}q^{-\omega_0} = \sqrt{s}Q^{-\omega_0(\delta-2\varepsilon)} \leq Q^{-\omega_0(\delta-3\varepsilon)} < Q^{-1-\varepsilon},$$

and

$$|\lambda| \leq 2q = 2Q^{(\delta-2\varepsilon)} \leq Q^{(\delta-\varepsilon)}.$$

Since q can be taken arbitrarily large, Q, can also be taken so large that [Fis15, Proposition 1 Part b)] applies. Then the above two inequalities and that fact that $\underline{v} = (q, p_1, \ldots, p_s) \neq \underline{0}$ give a contradiction. Hence, we obtain that

$$\omega(\underline{\xi}) \leq \frac{1}{\delta}.$$

From now on, we focus on the case $s = 2$. Our proof of Theorem 1.5 is based on the work of Bugeaud, Han, Wen and Yao [BHWY16]. As is the case with their approach, we will be using Padé approximants, to be precise, their higher-dimensional analogs. These are, roughly speaking, rational best approximations to series. Let us introduce the following notation. For any $f \in \mathbb{Q}[[z]]$ and $k \in \mathbb{N}$ we write

$$f(z) = O(z^k)$$

if

$$f(z) = \sum_{n \geq k} c_n z^j \quad \text{with } c_n \in \mathbb{Q},$$

that is, if all the terms of f of order strictly less than k vanish.

In dimension 1 Padé approximants are defined as follows:

Definition 2.5 (Padé approximant)**.** Let $f \in \mathbb{Q}[[z]]$. A Padé approximant of order k to f is a reduced rational fraction $P(z)/Q(z) \in \mathbb{Q}(z)$ such that $\deg P \leq k-1$, $\deg Q \leq k$, and

$$f(z) - \frac{P(z)}{Q(z)} = O(z^{2k}).$$

The higher-dimensional analogs of Padé approximants are known in the literature as Hermite-Padé approximants [KVAW05] or Hermite-Padé polynomials [Sue18]. As for real vectors, one may consider two types of approximation: simultaneous and dual. This motivates the notion of Type-I and Type-II Hermite-Padé approximants, where Type-I approximants correspond to dual approximation and Type-II approximants to simultaneous approximation. For us it will be important to work with Type I Hermite-Padé approximants, which we now proceed to define.

Definition 2.6 (Type I Hermite–Padé approximant)**.** Let

$$f_1(z) = \sum_{n \geq 0} c_n z^n, \qquad f_2(z) = \sum_{n \geq 0} d_n z^n$$

be two formal power series with coefficients in $\mathbb{Q}$. Fix an integer $k \geq 1$.

A Type I Hermite–Padé approximant of order k to (f_1, f_2) is a triple of polynomials

$$(P_0(z), P_1(z), P_2(z)) \in \mathbb{Q}[z]^3$$

such that

$$\deg P_0 \le k_0, \qquad \deg P_1 \le k_1, \qquad \deg P_2 \le k_2,$$

with $k_0 + k_1 + k_2 = 3k$, and

$$P_0(z) + P_1(z)f_1(z) + P_2(z)f_2(z) = O(z^{3k+2}). \tag{2.5}$$

We call the triple $(k_0, k_1, k_2) \in \mathbb{N}^3$ the multi-index of the approximant [Sue18, LP15, SR22].

Remark 2.7. Let us briefly explain why $3k + 2$ is the correct exponent in (2.5). First, observe that, for any choice of the polynomials P_1 and P_2, the conditions $[z^i](P_0(z) + P_1(z)f_1(z) + P_2(z)f_2(z)) = 0$ for $i = 0, \dots, k_0$ may be obtained by selecting the coefficients of the polynomial P_0 appropriately. To annihilate terms of degree $> k_0$ in the expression $(P_0(z) + P_1(z)f_1(z) + P_2(z)f_2(z))$, one is required to solve a system of linear equations, where the coefficients of P_1 and P_2 are the unknowns. There are in total $(k_1 + 1) + (k_2 + 1) = 3k - k_0 + 2$ unknowns and, since the constant terms in the system are null, we are guaranteed to find a non-trivial solution if the system contains at most $3k - k_0 + 1$ equations. Thus, by choosing P_1 and P_2 appropriately, we can annihilate all terms up to degree $3k + 1$,, explaining the term $O(z^{3k+2})$ on the right-hand side of (2.5). Note that for general vectors of dimension s, the same reasoning applies and we can annihilate all terms in the expression $(P_0(z) + P_1(z)f_1(z) + P_2(z)f_2(z) + \cdots + P_s(z)f_s(z))$ up to degree $k_0 + [(s+1)k - k_0 + s - 1] = (s+1)k + s - 1$, showing that in the definition of Hermite-Padé Approximant for general s the condition $(P_0(z) + P_1(z)f_1(z) + P_2(z)f_2(z) + \cdots + P_s(z)f_s(z)) = O(z^{(s+1)k+s})$ should be imposed.

We now proceed to prove the following lemma, which, for each large enough degree k, produces three special triples of type I Hermite–Padé approximants of order k. These will be instrumental later in our argument. We use the following notation. For each $k \in \mathbb{N}$ and $P \in \mathbb{Q}[z]$ we denote by $[z^k]P(z)$ the coefficient of the monomial of degree k of the polynomial P.

The proof of the lemma is inspired by [KVAW05, Theorem 5.1].

Lemma 2.8. *Let*

$$f_1(z) = \sum_{n \ge 0} c_n z^n, \qquad f_2(z) = \sum_{n \ge 0} d_n z^n$$

be non-trivial formal power series over $\mathbb{Q}$. Fix $k \ge 1$ large enough such that at least one of $c_0, \dots, c_{2k}$ is nonzero and at least one of $d_0, \dots, d_{2k}$ is nonzero. Assume that

$$H_k(f_1, f_2) \neq 0.$$

Then there exist unique triples

$$\underline{P}_j = (P_{j,0}, P_{j,1}, P_{j,2}) \in \mathbb{Q}[z]^3 \quad \text{for } j = 0, 1, 2,$$

with the following properties:

- $L_j(z) := P_{j,0}(z) + P_{j,1}(z)f_1(z) + P_{j,2}(z)f_2(z) = O(z^{3k+2})$ *for* $j = 0, 1, 2$.
- P_0 *is of multi-index* (k, k, k) *and* $[z^k]P_{0,0}(z) = 1$*;*

- P_1 *is of multi-index* $(k-1, k+1, k)$ *and* $[z^{k+1}]P_{1,1}(z) = 1$*;*
- P_2 *is of multi-index* $(k-1, k, k+1)$ *and* $[z^{k+1}]P_{2,2}(z) = 1$.

Moreover,

$$\det \begin{pmatrix} P_{0,0}(z) & P_{0,1}(z) & P_{0,2}(z) \\ P_{1,0}(z) & P_{1,1}(z) & P_{1,2}(z) \\ P_{2,0}(z) & P_{2,1}(z) & P_{2,2}(z) \end{pmatrix} = z^{3k+2}.$$

Proof. We first prove the existence and uniqueness of the three triples. Consider a triple $(P_{0,0}, P_{0,1}, P_{0,2}) \in \mathbb{Q}[z]^3$ of polynomials (whose coefficients are yet to be determined) with

$$\deg P_{0,0} \le k, \qquad \deg P_{0,1}, \deg P_{0,2} \le k, \qquad [z^k]P_{0,0}(z) = 1.$$

We write

$$P_{0,0}(z) = z^k + \widetilde{P}_{0,0}(z),$$

where $\deg \widetilde{P}_{0,0} \le k-1$, and

$$P_{0,1}(z) = \sum_{j=0}^{k} a_j z^j, \qquad P_{0,2}(z) = \sum_{j=0}^{k} b_j z^j.$$

If the condition

$$L_0(z) = P_{0,0}(z) + P_{0,1}(z)f_1(z) + P_{0,2}(z)f_2(z) = O(z^{3k+2})$$

holds, it is clear that the coefficients of $\widetilde{P}_{0,0}$ of order strictly less than k are uniquely determined by $P_{0,1}$ and $P_{0,2}$. Conversely, for degrees $k, k+1, \dots, 3k+1$, the following system of equations must be satisfied:

$$\begin{cases} a_k c_0 + \cdots + a_0 c_k + b_k d_0 + \cdots + b_0 d_k = -1, \\ a_k c_1 + \cdots + a_0 c_{k+1} + b_k d_1 + \cdots + b_0 d_{k+1} = 0, \\ \vdots \\ a_k c_{2k+1} + \cdots + a_0 c_{3k+1} + b_k d_{2k+1} + \cdots + b_0 d_{3k+1} = 0. \end{cases}$$

Note that the associated determinant is exactly $H_k(f_1, f_2)$ and, since we assumed that $H_k(f_1, f_2) \neq 0$, this system has a unique solution. This proves the existence of the triple $\underline{P}_0$.

Next, we move to $\underline{P}_1$. We consider polynomials $(P_{1,0}, P_{1,1}, P_{1,2}) \in \mathbb{Q}[z]^3$ whose coefficients are yet to be determined and such that

$$\deg P_{1,0} \le k-1, \qquad \deg P_{1,1} \le k+1, \qquad \deg P_{1,2} \le k, \qquad [z^{k+1}]P_{1,1}(z) = 1.$$

Write

$$P_{1,1}(z) = z^{k+1} + \sum_{j=0}^{k} a_j z^j, \qquad P_{1,2}(z) = \sum_{j=0}^{k} b_j z^j.$$

Once again, if

$$L_1(z) = P_{1,0}(z) + P_{1,1}(z)f_1(z) + P_{1,2}(z)f_2(z) = O(z^{3k+2}),$$

holds, the coefficients of $P_{1,0}$ are uniquely determined by the coefficients of $P_{1,1}$ and $P_{1,2}$, and the coefficients of these two polynomials must satisfy the linear system

$$\begin{cases} a_k c_0 + \cdots + a_0 c_k + b_k d_0 + \cdots + b_0 d_k = 0, \\ a_k c_1 + \cdots + a_0 c_{k+1} + b_k d_1 + \cdots + b_0 d_{k+1} = -c_0, \\ \vdots \\ a_k c_{2k+1} + \cdots + a_0 c_{3k+1} + b_k d_{2k+1} + \cdots + b_0 d_{3k+1} = -c_{2k}, \end{cases}$$

where we used the condition $[z^{k+1}]P_{1,1}(z) = 1$. The determinant of this system is again $H_k(f_1, f_2)$ and this again produces unique polynomials $P_{1,1}, P_{1,2}$. The construction of $\underline{P}_2$ is essentially identical to that of $\underline{P}_1$.

Now, define

$$\Delta(z) = \det \begin{pmatrix} P_{0,0}(z) & P_{0,1}(z) & P_{0,2}(z) \\ P_{1,0}(z) & P_{1,1}(z) & P_{1,2}(z) \\ P_{2,0}(z) & P_{2,1}(z) & P_{2,2}(z) \end{pmatrix}.$$

In the determinant expansion, the only term with of degree $3k+2$ is $P_{0,0}(z)P_{1,1}(z)P_{2,2}(z)$. Therefore,

$$\deg \Delta(z) \le 3k+2, \quad \text{and} \quad [z^{3k+2}]\Delta(z) = 1. \tag{2.6}$$

Meanwhile, recalling that

$$L_j(z) = P_{j,0}(z) + P_{j,1}(z)f_1(z) + P_{j,2}(z)f_2(z),$$

we may apply a column operation to see that

$$\Delta(z) = \det \begin{pmatrix} L_0(z) & P_{0,1}(z) & P_{0,2}(z) \\ L_1(z) & P_{1,1}(z) & P_{1,2}(z) \\ L_2(z) & P_{2,1}(z) & P_{2,2}(z) \end{pmatrix}.$$

Expanding the determinant along the first column, and using

$$L_j(z) = O(z^{3k+2}),$$

we obtain

$$\Delta(z) = O(z^{3k+2}). \tag{2.7}$$

Thus, (2.6) and (2.7) imply

$$\Delta(z) = z^{3k+2}.$$

This proves the lemma. □

Remark 2.9. In a similar manner, one can also construct $s+1$ independent Type I Padé approximants $(P_0(z), P_1(z), \cdots, P_s(z)) \in \mathbb{Q}[z]^{s+1}$ for $(f_1, \cdots, f_s)$, of a sufficiently large order k.

3. Proof of Theorem 1.5

Recall that we are given a pair of polynomials

$$\underline{f}(z) = (f_1(z), f_2(z)) \in \mathbb{Z}[[z]]^2$$

such that both f_1 and f_2 converge inside the unit disk, and such that $\underline{f}$ satisfies the Mahler system

$$\underline{f}(z) = \frac{C(z)}{D(z)}\underline{f}(z^d) + \frac{A(z)}{B(z)},$$

where

$$B(z), D(z) \in \mathbb{Q}[z], \quad C(z) \in (\mathbb{Q}[z])^{2\times 2}, \quad \text{and} \quad A(z) \in (\mathbb{Q}[z])^{2\times 1}.$$

We fix an integer $b \geq 1$ and we assume that

$$B\left(\frac{1}{b^{d^m}}\right) D\left(\frac{1}{b^{d^m}}\right) \det C\left(\frac{1}{b^{d^m}}\right) \neq 0 \quad \text{for all } m \geq 0.$$

First we observe that f_1 and f_2 satisfy the linear Mahler system

$$\begin{pmatrix} 1 \\ f_1(z) \\ f_2(z) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ A_1(z)/B(z) & C_{1,1}(z)/D(z) & C_{1,2}(z)/D(z) \\ A_2(z)/B(z) & C_{2,1}(z)/D(z) & C_{2,2}(z)/D(z) \end{pmatrix} \begin{pmatrix} 1 \\ f_1(z^d) \\ f_2(z^d) \end{pmatrix}.$$

Assume by contradiction that $1, f_1(1/b), f_2(1/b)$ are not linearly independent over $\mathbb{Q}$. Then, according to [AF17, Theorem 1.4], there exists a homogeneous linear polynomial

$$Q(z, X_0, X_1, X_2) = q_0(z)X_0 + q_1(z)X_1 + q_2(z)X_2 \in \overline{\mathbb{Q}}[z, X_0, X_1, X_2]$$

such that

$$Q(z, 1, f_1(z), f_2(z)) = 0. \tag{3.1}$$

Yet by Lemma 2.1 and our assumption on the Hankel determinants in Theorem 1.5, we see that $1, f_1(z), f_2(z)$ are linearly independent over $\overline{\mathbb{Q}}(z)$ and thus they cannot satisfy (3.1). This is a contradiction and hence the numbers $1, f_1(1/b), f_2(1/b)$ are linearly independent over $\mathbb{Q}$.

We now proceed as in [BHWY16]. Iterating the Mahler system, we find that for every $m \geq 1$

$$\underline{f}(z) = \frac{C_m(z)}{d_m(z)}\underline{f}(z^{d^m}) + \frac{A_m(z)}{b_m(z)}, \tag{3.2}$$

where

$$\begin{aligned} C_m(z) &= C(z)C(z^d)\cdots C(z^{d^{m-1}}), \\ d_m(z) &= D(z)D(z^d)\cdots D(z^{d^{m-1}}), \\ b_m(z) &= d_{m-1}(z)\prod_{r=0}^{m-1} B(z^{d^r}), \end{aligned} \tag{3.3}$$

and

$$A_m(z) = \sum_{j=0}^{m-1} \left(C_j(z)\, A(z^{d^j}) \prod_{\substack{0\le r\le m-1 \\ r\neq j}} B(z^{d^r}) \cdot \prod_{r=j}^{m-2} D(z^{d^r}) \right), \tag{3.4}$$

with the convention that

$$\prod_{r=m-1}^{m-2} D(z^{d^r}) = 1.$$

By assumption we have that

$$b_m(1/b) d_m(1/b) \det C_m(1/b) \neq 0.$$

Fix i such that n_i is large enough. Since $H_{n_i}(f_1, f_2) \neq 0$, by Lemma 2.8, we can construct three triples $\underline{P}_{i,j}$ $(j = 0, 1, 2)$ such that

$$P_{i,j,0}(z) + P_{i,j,1}(z) f_1(z) + P_{i,j,2}(z) f_2(z) = O(z^{3n_i+2}) \quad \text{for } j = 0, 1, 2, \tag{3.5}$$

with

$$\max_{0\le r\le 2} \deg P_{i,j,r} \le n_i + 1,$$

and

$$\det \begin{pmatrix} P_{i,0,0}(z) & P_{i,0,1}(z) & P_{i,0,2}(z) \\ P_{i,1,0}(z) & P_{i,1,1}(z) & P_{i,1,2}(z) \\ P_{i,2,0}(z) & P_{i,2,1}(z) & P_{i,2,2}(z) \end{pmatrix} = z^{3n_i+2}.$$

Now, for each $j = 0, 1, 2$, we substitute z^{d^m} into (3.5) and obtain

$$P_{i,j,0}(z^{d^m}) + (P_{i,j,1}(z^{d^m}), P_{i,j,2}(z^{d^m}))\underline{f}(z^{d^m}) = O\left(z^{d^m(3n_i+2)}\right). \tag{3.6}$$

Since $\det C_m(z) \neq 0$, we may invert (3.2) and write

$$\begin{aligned} \underline{f}(z^{d^m}) &= d_m(z) C_m(z)^{-1} \left(\underline{f}(z) - \frac{A_m(z)}{b_m(z)} \right) \\ &= \frac{d_m(z)}{\det C_m(z)} \operatorname{adj} C_m(z) \left(\underline{f}(z) - \frac{A_m(z)}{b_m(z)} \right). \end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7) and multiplying by $b_m(z) \det C_m(z)$, we conclude that

$$\mathcal{C}_{i,j,m}(z) + \mathcal{A}_{i,j,m}(z) f_1(z) + \mathcal{B}_{i,j,m}(z) f_2(z) = b_m(z) \det C_m(z) O\left(z^{d^m(3n_i+2)}\right), \tag{3.8}$$

where

$$(\mathcal{A}_{i,j,m}(z), \mathcal{B}_{i,j,m}(z)) := b_m(z) d_m(z) (P_{i,j,1}(z^{d^m}), P_{i,j,2}(z^{d^m})) \operatorname{adj} C_m(z), \tag{3.9}$$

and

$$\mathcal{C}_{i,j,m}(z) := -d_m(z)(P_{i,j,1}(z^{d^m}), P_{i,j,2}(z^{d^m})) \operatorname{adj} C_m(z) A_m(z) + b_m(z) \det C_m(z) P_{i,j,0}(z^{d^m}). \tag{3.10}$$

Evaluating (3.8) at $z = 1/b$, we obtain three relations

$$\begin{aligned} &\mathcal{C}_{i,j,m}(1/b) + \mathcal{A}_{i,j,m}(1/b)\xi_1 + \mathcal{B}_{i,j,m}(1/b)\xi_2 \\ &\qquad = b_m(1/b) \det C_m(1/b) O\left(b^{-d^m(3n_i+2)}\right) \quad \text{for } j = 0, 1, 2 \end{aligned} \tag{3.11}$$

where

$$\xi_1 = f_1(1/b), \qquad \xi_2 = f_2(1/b),$$

valid for any i and m, with i large enough. Note that the notation O is used here in the sense of Bachmann-Landau and that the implicit constant is absolute and only depends on f_1 and f_2.

Since $B, D, \det C$ are fixed polynomials, they are bounded on the closed unit disk. Thus there exists $K_0 > 0$ such that for $|z| \le 1$

$$|B(z)|,\ |D(z)|,\ |\det C(z)| \le K_0.$$

Then for all $m \ge 0$

$$\left|B\left(\frac{1}{b^{d^m}}\right)\right|, \left|D\left(\frac{1}{b^{d^m}}\right)\right|, \left|\det C\left(\frac{1}{b^{d^m}}\right)\right| \le K_0.$$

By this and (3.3), we see that for all $m \ge 0$

$$|b_m(1/b) \det C_m(1/b)| \le K_0^{3m}.$$

Let $K_1 > 0$ be a fixed constant and observe that for any i there exists $N_i > 0$ such that for all $m > N_i$

$$K_1 b^{-d^m(3n_i+2)} \le b^{-d^m(3n_i+2)+d^m},$$

and

$$|b_m(1/b) \det C_m(1/b)| \le K_0^{3m} \le b^{d^m}.$$

Then (3.11) implies that for all large enough i and all $m > N_i$ it holds that

$$|\mathcal{C}_{i,j,m}(1/b) + \mathcal{A}_{i,j,m}(1/b)\xi_1 + \mathcal{B}_{i,j,m}(1/b)\xi_2| \le b^{-3n_i d^m}. \tag{3.12}$$

Our goal is now to multiply both sides of (3.12) by a large integer, in order to neutralize denominators on the left-hand side. To do so, we first estimate the degrees of $\mathcal{A}_{i,j,m}(z), \mathcal{B}_{i,j,m}(z), \mathcal{C}_{i,j,m}(z)$.

Lemma 3.1. *Let*

$$\kappa := \max_{a=1,2} \deg A_a(z), \qquad \lambda := \deg B(z), \qquad \mu := \max_{1\le a,b\le 2} \deg C_{a,b}(z), \qquad \nu := \deg D(z),$$

and let $\Gamma = \kappa + \lambda + 2\mu + 2\nu$. *Then for any* $j = 0, 1, 2$ *it holds that*

$$\deg \mathcal{A}_{i,j,m}(z), \deg \mathcal{B}_{i,j,m}(z), \deg \mathcal{C}_{i,j,m}(z) \le (n_i + 1 + \Gamma)d^m.$$

Proof. In this proof we will repeatedly use (3.3) and (3.4). We begin by observing that

$$\max_{1\le a,b\le 2} \deg(C_m)_{a,b}(z) \le \sum_{j=0}^{m-1} \mu d^j = \mu \frac{d^m - 1}{d-1} \le \mu d^m,$$

and

$$\deg d_m(z) \le \sum_{j=0}^{m-1} \nu d^j \le \nu d^m.$$

Moreover,

$$\deg b_m(z) \le \deg d_{m-1}(z) + \sum_{j=0}^{m-1} \deg B(z^{d^j}) \le (\nu+\lambda)d^m,$$

so that, by (3.4),

$$\max_{i=1,2} \deg(A_m)_a(z) \le (\kappa+\lambda+\mu+\nu)d^m.$$

Since $C_m(z)$ is a 2×2 matrix, the entries of $\operatorname{adj} C_m(z)$ are, up to signs, entries of $C_m(z)$. Hence,

$$\max_{1\le a,b\le 2} \deg\big(\operatorname{adj} C_m(z)\big)_{a,b}(z) \le \mu d^m.$$

Recall that

$$\deg P_{i,j,0}, \deg P_{i,j,1}, \deg P_{i,j,2} \le n_i + 1.$$

This gives

$$\deg P_{i,j,r}(z^{d^m}) \le (n_i+1)d^m \quad \text{for } r = 0,1,2.$$

Now, recall that

$$(\mathcal{A}_{i,j,m}(z), \mathcal{B}_{i,j,m}(z)) = b_m(z)d_m(z)(P_{i,j,1}(z^{d^m}), P_{i,j,2}(z^{d^m}))\operatorname{adj} C_m(z).$$

Then

$$\begin{aligned}\deg \mathcal{A}_{i,j,m}(z), \deg \mathcal{B}_{i,j,m}(z) &\le (\nu+\lambda)d^m + \nu d^m + (n_i+1)d^m + \mu d^m \\ &= (n_i+1+\lambda+\mu+2\nu)d^m \le (n_i+1+\Gamma)d^m.\end{aligned}$$

For $C_{i,j,m}$, on the other hand, we have

$$\mathcal{C}_{i,j,m}(z) = -d_m(z)(P_{i,j,1}(z^{d^m}), P_{i,j,2}(z^{d^m}))\operatorname{adj} C_m(z)A_m(z) + b_m(z)\det C_m(z)\, P_{i,j,0}(z^{d^m}).$$

The degree of the first term is bounded above by

$$\nu d^m + (n_i+1)d^m + \mu d^m + (\kappa+\lambda+\mu+\nu)d^m.$$

For the second term, we use

$$\deg\det C_m(z) \le 2\mu d^m,$$

whence

$$\deg\Big(b_m(z)\det C_m(z)P_{j,0}(z^{d^m})\Big) \le (\nu+\lambda)d^m + 2\mu d^m + (n_i+1)d^m.$$

Thus,

$$\deg \mathcal{C}_{i,j,m}(z) \le (n_i+1+\kappa+\lambda+2\mu+2\nu)d^m \le (n_i+1+\Gamma)d^m.$$

This concludes the proof. □

Our next step is the following.

Lemma 3.2. *For all large enough i there exists a constant $M_i > 0$ such that for each $m \ge M_i$ one may find an integer $D_{i,m} \in \mathbb{Z}\setminus\{0\}$ with*

$$|D_{i,m}| \le b^{(n_i+2+\Gamma)d^m}$$

for which

$$D_{i,m}\mathcal{A}_{i,j,m}(1/b), D_{i,m}\mathcal{B}_{i,j,m}(1/b), D_{i,m}\mathcal{C}_{i,j,m}(1/b) \in \mathbb{Z}$$

for all $j = 0,1,2$. Moreover, for any $m \ge M_i$ it holds that

$$|\mathcal{A}_{i,j,m}(1/b)|, |\mathcal{B}_{i,j,m}(1/b)|, |\mathcal{C}_{i,j,m}(1/b)| \le b^{d^m}.$$

Proof. Let $K_{2,i}$ be a common denominator to all coefficients of the polynomials $A_a, B, C_{a,b}$, D, and $P_{i,j,r}$, and let $K_{3,i}$ be a bound on their values within the unit disk. By (3.9) and (3.10), we have that the coefficients of $\mathcal{A}_{i,j,m}, \mathcal{B}_{i,j,m}, \mathcal{C}_{i,j,m}$ have common denominator dividing $K_{2,i}^{100m}$ and

$$|\mathcal{A}_{i,j,m}(1/b)|, |\mathcal{B}_{i,j,m}(1/b)|, |\mathcal{C}_{i,j,m}(1/b)| \le K_{3,i}^{100m}.$$

For any fixed value of i, let M_i be so large that for any $m \ge M_i$ it holds that

$$K_{2,i}^{100m}, K_{3,i}^{100m} \le b^{d^m}.$$

Put

$$D_{i,m} := K_{2,i}^{100m} b^{(n_i+1+\Gamma)d^m} \le b^{(n_i+2+\Gamma)d^m}.$$

Thanks to Lemma 3.1, we have that

$$D_{i,m}\mathcal{A}_{i,j,m}(1/b), D_{i,m}\mathcal{B}_{i,j,m}(1/b), D_{i,m}\mathcal{C}_{i,j,m}(1/b) \in \mathbb{Z}$$

for all $j = 0, 1, 2$, and

$$|\mathcal{A}_{i,j,m}(1/b)|, |\mathcal{B}_{i,j,m}(1/b)|, |\mathcal{C}_{i,j,m}(1/b)| \le K_{3,i}^{100m} \le b^{d^m}.$$

The claim follows. □

Now, fix i. Let $D_{i,m}$ be as in Lemma 3.2 and set

(3.13)
$$\alpha_{i,j,m} = D_{i,m}\mathcal{A}_{i,j,m}(1/b), \quad \beta_{i,j,m} = D_{i,m}\mathcal{B}_{i,j,m}(1/b), \quad \text{and } \gamma_{i,j,m} = D_{i,m}\mathcal{C}_{i,j,m}(1/b),$$

so that

$$\alpha_{i,j,m}, \beta_{i,j,m}, \gamma_{i,j,m} \in \mathbb{Z}.$$

Put

$$L_{i,j,m}(\underline{\xi}) = \gamma_{i,j,m} + \alpha_{i,j,m}\xi_1 + \beta_{i,j,m}\xi_2.$$

Combining (3.12) and Lemma 3.2, we see that for all $m \ge \max\{N_i, M_i\}$ it holds that

$$|L_{i,j,m}(\underline{\xi})| \le D_{i,m} b^{-3n_i d^m} \le b^{-(2n_i-2-\Gamma)d^m} \tag{3.14}$$

$$\max\{|\alpha_{i,j,m}|, |\beta_{i,j,m}|, |\gamma_{i,j,m}|\} \le b^{(n_i+3+\Gamma)d^m}. \tag{3.15}$$

Now let

$$\mathcal{H}_{i,m} = b^{(n_i+3+\Gamma)d^m}.$$

and

$$\delta_i = \frac{2n_i - 2 - \Gamma}{n_i + 3 + \Gamma}.$$

Then we may rewrite (3.14) and (3.15) as

$$|L_{i,j,m}(\underline{\xi})| \le \mathcal{H}_{i,m}^{-\delta_i} \tag{3.16}$$

$$\max\{|\alpha_{i,j,m}|, |\beta_{i,j,m}|, |\gamma_{i,j,m}|\} \le \mathcal{H}_{i,m}, \tag{3.17}$$

where the inequalities hold for all $j = 0, 1, 2$. One could hope to apply Lemma 2.8, fixing m and letting i tend to infinity. In this case, δ_i approaches 2 and

$$\mathcal{H}_{i+1,m} = \mathcal{H}_{i,m}^{(n_{i+1}+3+\Gamma)/(n_i+3+\Gamma)},$$

showing that θ is approximately equal to ρ. Lemma (2.8) would then give an upper bound of the form $\omega^*(\underline{\xi}) \leq 2\rho$. However, this strategy does not work, since as i increases m must also increase, if we are to use (3.12) and Lemma 3.2. We therefore need to adopt a more subtle approach. Once again, we are inspired by the proof of [BHWY16, Theorem 4.1]. The key input is to let i range in a bounded interval and m diverge. Let us make this precise.

Fix $i_0 \in \mathbb{N}$ and for any integer ℓ define

$$\mathcal{A}_\ell = \left\{ i \geq i_0 : \ d^{\ell-1} \leq n_i + 3 + \Gamma < d^\ell \right\}.$$

Assume $\mathcal{A}_\ell \neq \emptyset$, and write

$$\mathcal{A}_\ell = \{i_1 < i_2 < \cdots < i_{t_\ell}\}.$$

For $1 \leq t \leq t_\ell$ and $m \geq M_\ell$ (where M_ℓ is a strictly increasing sequence to be chosen later) consider

$$\mathcal{H}_{i_t,m} = b^{(n_{i_t}+3+\Gamma)d^m}.$$

Arrange the distinct numbers $\mathcal{H}_{i_t,m}$, with $1 \leq t \leq t_\ell$ and $m \geq M_\ell$, in increasing order, and denote them by:

$$h_{\ell,1} < h_{\ell,2} < h_{\ell,3} < \cdots .$$

Then the following holds.

Lemma 3.3. *Let $\eta > 0$. Then there exists a constant $\ell_0 = \ell_0(\eta, \rho)$ such that for all $\ell \geq \ell_0$ for which $\mathcal{A}_\ell \neq \emptyset$ it holds that*

$$h_{\ell,r+1} \leq h_{\ell,r}^{\rho^2+\eta}$$

for all $r \in \mathbb{N}$.

Proof. Fix r and write $h_{\ell,r} = \mathcal{H}_{i_t,m}$. There are two cases. First suppose $t < t_\ell$. Then $\mathcal{H}_{i_{t+1},m}$ is one of the numbers in the sequence $h_{\ell,1}, h_{\ell,2}, \ldots$ larger than $h_{\ell,r} = \mathcal{H}_{i_t,m}$ and

$$h_{\ell,r+1} \leq \mathcal{H}_{i_{t+1},m}.$$

Therefore, if $\ell \geq \ell_0$ and ℓ_0 is sufficiently large

$$\frac{\log h_{\ell,r+1}}{\log h_{\ell,r}} \leq \frac{(n_{i_{t+1}} + 3 + \Gamma)d^m}{(n_{i_t} + 3 + \Gamma)d^m} \leq \rho + \eta \leq \rho^2 + \eta,$$

where we used the fact that $\rho \geq 1$. Now, suppose $t = t_\ell$. Since $d(n_{i_1} + 3 + \Gamma) \geq d^\ell > n_{i_t} + 3 + \Gamma$,

$$h_{\ell,r+1} \leq \mathcal{H}_{i_1,m+1} = b^{(n_{i_1}+3+\Gamma)d^{m+1}}.$$

Therefore,

$$\frac{\log h_{\ell,r+1}}{\log h_{\ell,r}} \leq \frac{(n_{i_1} + 3 + \Gamma)d^{m+1}}{(n_{i_t} + 3 + \Gamma)d^m}. \tag{3.18}$$

Now fix $0 < \eta' < \eta$. Since i_1 is the first index in $\mathcal{A}_\ell$, we have that $n_{i_1-1} + 3 + \Gamma < d^{\ell-1}$ and if ℓ_0 is sufficiently large, for all $\ell \geq \ell_0$

$$n_{i_1} + 3 + \Gamma \leq (\rho + \eta')(n_{i_1-1} + 3 + \Gamma) < (\rho + \eta')d^{\ell-1}, \tag{3.19}$$

where we used the fact that if ℓ is large so is i_1. Since $t = t_\ell$ is the last index in $\mathcal{A}_\ell$, we also have that

$$n_{i_t+1} + 3 + \Gamma \geq d^\ell,$$

and, if ℓ_0 is sufficiently large,

$$n_{i_t} + 3 + \Gamma \geq (\rho + \eta')^{-1}(n_{i_{t+1}} + 3 + \Gamma) \geq (\rho + \eta')^{-1}d^\ell. \tag{3.20}$$

Combining (3.18), (3.19), and (3.20), we conclude that

$$\frac{\log h_{\ell,r+1}}{\log h_{\ell,r}} \leq \frac{(n_{i_1} + 3 + \Gamma)d^{m+1}}{(n_{i_t} + 3 + \Gamma)d^m} \leq \frac{(\rho + \eta')d^{\ell-1+m+1}}{(\rho + \eta')^{-1}d^{\ell+m}} = (\rho + \eta')^2.$$

Taking $\eta' = \eta/(3\rho)$, the calim is proved. □

Now, fix $\eta > 0$ and choose ℓ so large that for any $i \in \mathcal{A}_\ell$ we have

$$\delta_i = \frac{2n_i - 2 - \Gamma}{n_i + 3 + \Gamma} \geq 2 - \eta$$

and so that Lemma 3.3 holds. Choose also M_ℓ large enough to ensure that (3.14) and (3.15) hold. Relabel $h_{\ell,r}$ as h_r and denote the corresponding integers $\alpha_{i,j,m}, \beta_{i,j,m}, \gamma_{i,j,m}$ by $\alpha_{r,j}, \beta_{r,j}, \gamma_{r,j}$ for $j = 0, 1, 2$. Then (3.14) and (3.15) become

$$|\gamma_{r,j} + \alpha_{r,j}\xi_1 + \beta_{r,j}\xi_2| \leq h_r^{-2+\eta} \tag{3.21}$$

$$\max\{|\alpha_{r,j}|, |\beta_{r,j}|, |\gamma_{r,j}|\} \leq h_r. \tag{3.22}$$

Lemma 3.3 also gives

$$h_{r+1} \leq h_r^{\rho^2+\eta}. \tag{3.23}$$

Thus, (3.21), (3.22), and (3.23) are enough to invoke Lemma 2.2, provided we show the following.

Lemma 3.4. *For any sufficiently large i and any $m \geq 0$, it holds that*

$$\det \begin{pmatrix} \gamma_{i,0,m} & \gamma_{i,1,m} & \gamma_{i,2,m} \\ \alpha_{i,0,m} & \alpha_{i,1,m} & \alpha_{i,2,m} \\ \beta_{i,0,m} & \beta_{i,1,m} & \beta_{i,2,m} \end{pmatrix} \neq 0.$$

Assume that Lemma 3.4 holds. Then, by Lemma 2.2, we have that for any $\eta > 0$

$$\omega^*(\underline{\xi}) \leq \frac{2(\rho^2 + \eta)}{1 - \eta}.$$

Since η is arbitrarily small, the claim follows.

Let us now conclude by proving Lemma 3.4.

Proof. recall that $\alpha_{i,j,m}, \beta_{i,j,m}$, and $\gamma_{i,j,m}$ are defined in (3.13). Since $D_{i,m} \neq 0$, we simply need to study $\det \mathcal{M}_{i,z}(1/b)$, where

$$\mathcal{M}_{i,m}(z) = \begin{pmatrix} \mathcal{C}_{i,0,m}(z) & \mathcal{A}_{i,0,m}(z) & \mathcal{B}_{i,0,m}(z) \\ \mathcal{C}_{i,1,m}(z) & \mathcal{A}_{i,1,m}(z) & \mathcal{B}_{i,1,m}(z) \\ \mathcal{C}_{i,2,m}(z) & \mathcal{A}_{i,2,m}(z) & \mathcal{B}_{i,2,m}(z) \end{pmatrix}.$$

Then, by (3.9) and (3.10), we have that

$$\mathcal{M}_{i,m}(z) = P_i(z^{d^m}) T_m(z),$$

where

$$P_i(z) = \begin{pmatrix} P_{i,0,0}(z) & P_{i,0,1}(z) & P_{i,0,2}(z) \\ P_{i,1,0}(z) & P_{i,1,1}(z) & P_{i,1,2}(z) \\ P_{i,2,0}(z) & P_{i,2,1}(z) & P_{i,2,2}(z) \end{pmatrix},$$

with $P_{i,j,r}$ as in (3.5), and

$$T_m(z) = \begin{pmatrix} b_m(z) \det C_m(z) & \mathbf{0}^T \\ -d_m(z) \operatorname{adj} C_m(z) A_m(z) & b_m(z) d_m(z) \operatorname{adj} C_m(z) \end{pmatrix},$$

with C_m, b_m, and d_m as in (3.3). Since

$$\det T_m(z) = b_m(z)^3 d_m(z)^2 (\det C_m(z))^2,$$

we have that

$$\det \mathcal{M}_{i,m}(z) = z^{d^m(3n_i+2)} b_m(z)^3 d_m(z)^2 (\det C_m(z))^2.$$

Therefore, by (3.3) and our assumption in Theorem 1.5, we have that $\det \mathcal{M}_{i,m}(1/b) \neq 0$. □

4. Proof of Theorem 1.8

Let $\underline{f}(z)$ be a solution to the system

$$\underline{f}(z) = \begin{pmatrix} 1 & \alpha z \\ 1 & \beta z \end{pmatrix} \underline{f}(z^2) \tag{4.1}$$

and assume that $\underline{f}$ is as in (1.3). It is easy to see that the coefficients of f_1 and f_2 must satisfy the following recursive relations:

$$\begin{aligned} &c_{2n} = c_n, \quad c_{2n+1} = \alpha d_n, \\ &d_{2n} = c_n, \quad d_{2n+1} = \beta d_n, \\ &d_0 = c_0 = 1. \end{aligned} \tag{4.2}$$

Let us first verify that $\underline{f}$ converges in the unit disk. Set

$$M := \max\{1, |\alpha|, |\beta|\}.$$

Then for every $n \geq 1$,

$$|c_n|, |d_n| \leq M^{\log_2 n + 1} = M n^{\log_2 M}.$$

Consequently, for every $z \in \mathbb{C}$ with $|z| < 1$,

$$\sum_{n \geq 0} |c_n z^n| \leq 1 + M \sum_{n \geq 1} n^{\log_2 M} |z|^n < \infty,$$

and similarly,

$$\sum_{n\geq 0} |d_n z^n| < \infty.$$

Hence, both $f_1(z)$ and $f_2(z)$ converge absolutely inside the open unit disk.

In order to apply Corollary 1.6, we further need to show that

$$B\left(\frac{1}{b^{d^m}}\right) D\left(\frac{1}{b^{d^m}}\right) \det C\left(\frac{1}{b^{d^m}}\right) \neq 0 \quad \text{for } m \geq 0.$$

In our case, $B = D = 1$ and $\det C(z) = (\beta - \alpha)z$. Since, $\alpha \neq \beta$, $\det C(z) \neq 0$ for any $z \neq 0$.

Our goal will be to prove that for all $n \geq 1$ the Hankel determinant $H_n = H_n(\underline{f})$ is not divisible by p, where p is the prime number in the statement of Theorem 1.8. This will show that $\rho = 1$ in (1.4), allowing us to conclude.

Let us denote by $H_n(\alpha, \beta)$ the Hankel determinants relative to the solution $\underline{f}$ to the system in (4.1), with $\underline{f}$ uniquely determined by the conditions (4.2). By our assumptions on α and β, we have that

$$H_n(\alpha, \beta) \equiv H_n(\alpha, 0) \pmod{p}.$$

Thus, in place of (4.2), we may instead consider sequences of integers $\tilde{c}_n, \tilde{d}_n$ satisfying the relations

$$\begin{aligned} &\tilde{c}_{2n} = \tilde{c}_n, \qquad \tilde{c}_{2n+1} = \alpha \tilde{d}_n, \\ &\tilde{d}_{2n} = \tilde{c}_n, \qquad \tilde{d}_{2n+1} = 0, \\ &\tilde{c}_0 = \tilde{d}_0 = 1, \end{aligned} \tag{4.3}$$

and it will suffice to show that none of the Hankel determinants for these new sequences are divisible by p for $n \geq 1$, to conclude that $H_n(\alpha, \beta) \neq 0$ for all $n \geq 1$. For $j > i, k > i$, let

$$C_{j,k}^{(i)} = \begin{pmatrix} \tilde{c}_i & \tilde{c}_{i+1} & \cdots & \tilde{c}_k \\ \tilde{c}_{i+1} & \tilde{c}_{i+2} & \cdots & \tilde{c}_{k+1} \\ \vdots & \vdots & & \vdots \\ \tilde{c}_j & \tilde{c}_{j+1} & \cdots & \tilde{c}_{j-i+k} \end{pmatrix}, \qquad D_{j,k}^{(i)} = \begin{pmatrix} \tilde{d}_i & \tilde{d}_{i+1} & \cdots & \tilde{d}_k \\ \tilde{d}_{i+1} & \tilde{d}_{i+2} & \cdots & \tilde{d}_{k+1} \\ \vdots & \vdots & & \vdots \\ \tilde{d}_j & \tilde{d}_{j+1} & \cdots & \tilde{d}_{j-i+k} \end{pmatrix}. \tag{4.4}$$

Then

$$H_n = H_n(\alpha, 0) = \det\begin{pmatrix} C_{2n+1,n}^{(0)} & D_{2n+1,n}^{(0)} \end{pmatrix}.$$

We define the following auxiliary determinants for $n \geq 1$:

$$\begin{aligned} P_n &= \det\begin{pmatrix} C_{2n,n}^{(0)} & D_{2n,n-1}^{(0)} \end{pmatrix}, \\ Q_n &= \det\begin{pmatrix} C_{2n+1,n}^{(1)} & D_{2n,n}^{(0)} \end{pmatrix}, \\ R_n &= \det\begin{pmatrix} C_{2n+2,n+1}^{(1)} & D_{2n+1,n}^{(0)} \end{pmatrix}, \qquad (n \geq 0) \\ T_n &= \det\begin{pmatrix} C_{2n,n}^{(1)} & D_{2n,n}^{(1)} \end{pmatrix}, \\ U_n &= \det\begin{pmatrix} C_{2n+1,n+1}^{(1)} & D_{2n+1,n}^{(1)} \end{pmatrix}. \end{aligned} \tag{4.5}$$

The key result of this section is the subsequent lemma, where we remove the tilde for convenience.

Lemma 4.1. *Let c_n and d_n for $n \geq 0$ be sequences of integers satisfying* (4.3)*, with $c_0 = d_0 = 1$, and let P_n, Q_n, R_n, T_n, U_n be as in* (4.5)*, with $C_{j,k}^{(i)}$ and $D_{j,k}^{(i)}$ as in* (4.4)*. Then the following relations hold for $n \geq 1$:*

$$|H_{2n}| = |\alpha^{2n+1} P_n Q_n|,$$

$$|H_{2n+1}| = |\alpha^{2n+2} H_n R_n|,$$

$$|P_{2n}| = |\alpha^{2n} P_n R_{n-1}|,$$

$$|P_{2n+1}| = |\alpha^{2n+2} H_n Q_n|,$$

$$|Q_{2n}| = |\alpha^{2n} P_n T_n|,$$

$$|Q_{2n+1}| = |\alpha^{2n+1} H_n U_n|,$$

$$|R_{2n}| = |\alpha^{2n} P_n U_n|,$$

$$|R_{2n+1}| = |\alpha^{2n+2} H_n T_{n+1}|,$$

$$|T_{2n}| = |\alpha^{2n} R_{n-1} T_n|,$$

$$|T_{2n+1}| = |\alpha^{2n+1} Q_n U_n|,$$

$$|U_{2n}| = |\alpha^{2n+1} Q_n T_n|,$$

$$|U_{2n+1}| = |\alpha^{2n+1} R_n U_n|.$$

Let us assume that Lemma 4.1 holds and show how to conclude that the numbers $H_n(\alpha, 0)$ are not divisible by p for any $n \geq 1$. Note that

$$|H_1| = |\alpha|^4, \qquad |P_1| = |Q_1| = |U_1| = |\alpha|^3,$$

$$|R_0| = |\alpha|, \qquad |R_1| = |\alpha|^5, \qquad |T_1| = |\alpha|^2.$$

Since every determinant on the right-hand side of the relations in Lemma 4.1 has a smaller index than the determinant on the left-hand side, and since $\alpha \neq 0$, we inductively have that $H_n, P_n, Q_n, R_n, T_n, U_n$ are all, up to their sign, powers of α. In particular, for all $n \geq 1$

$$H_n(\alpha, 0) = \sigma_n \alpha^{E_n},$$

where $\sigma_n \in \{-1, 1\}, E_n \in \mathbb{N}$. This shows that $H_n(\alpha, 0)$ is not divisible by p. Consequently, condition (1.4) holds with $\rho = 1$ and Theorem 1.5 implies that for any integer $b \geq 2$ it holds that

$$\mu(\underline{f}(1/b)) = \frac{3}{2}.$$

We devote the rest of this paper to the proof of Lemma 4.1.

Proof. In what follows, for a matrix A we denote by $|A|$ the absolute value of the determinant of A. First, we prove that $|H_{2n}| = |\alpha^{2n+1} P_n Q_n|$. Recall that

$$|H_{2n}| = \begin{vmatrix} c_0 & c_1 & \cdots & c_{2n} & d_0 & d_1 & \cdots & d_{2n} \\ c_1 & c_2 & \cdots & c_{2n+1} & d_1 & d_2 & \cdots & d_{2n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{4n} & c_{4n+1} & \cdots & c_{6n} & d_{4n} & d_{4n+1} & \cdots & d_{6n} \\ c_{4n+1} & c_{4n+2} & \cdots & c_{6n+1} & d_{4n+1} & d_{4n+2} & \cdots & d_{6n+1} \end{vmatrix}.$$

We exchange rows and columns so that the matrix turns into 8 blocks, where each block contains only even or odd index terms c_i and d_i :

$$|H_{2n}| = \begin{vmatrix} c_0 & c_2 & \cdots & c_{2n} & c_1 & \cdots & c_{2n-1} & d_0 & \cdots & d_{2n} & d_1 & \cdots & d_{2n-1} \\ c_2 & c_4 & \cdots & c_{2n+2} & c_3 & \cdots & c_{2n+1} & d_2 & \cdots & d_{2n+2} & d_3 & \cdots & d_{2n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{4n} & c_{4n+2} & \cdots & c_{6n} & c_{4n+1} & \cdots & c_{6n-1} & d_{4n} & \cdots & d_{6n} & d_{4n+1} & \cdots & d_{6n-1} \\ c_1 & c_3 & \cdots & c_{2n+1} & c_2 & \cdots & c_{2n} & d_1 & \cdots & d_{2n+1} & d_2 & \cdots & d_{2n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{4n+1} & c_{4n+3} & \cdots & c_{6n+1} & c_{4n+2} & \cdots & c_{6n} & d_{4n+1} & \cdots & d_{6n+1} & d_{4n+2} & \cdots & d_{6n} \end{vmatrix}.$$

Then we apply (4.3) and obtain

$$\begin{aligned} |H_{2n}| &= \begin{vmatrix} c_0 & \cdots & c_n & \alpha d_0 & \cdots & \alpha d_{n-1} & c_0 & \cdots & c_n & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{2n} & \cdots & c_{3n} & \alpha d_{2n} & \cdots & \alpha d_{3n-1} & c_{2n} & \cdots & c_{3n} & 0 & \cdots & 0 \\ \alpha d_0 & \cdots & \alpha d_n & c_1 & \cdots & c_n & 0 & \cdots & 0 & c_1 & \cdots & c_n \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \alpha d_{2n} & \cdots & \alpha d_{3n} & c_{2n+1} & \cdots & c_{3n} & 0 & \cdots & 0 & c_{2n+1} & \cdots & c_{3n} \end{vmatrix} \\ &= \begin{vmatrix} c_0 & \cdots & c_n & \alpha d_0 & \cdots & \alpha d_{n-1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{2n} & \cdots & c_{3n} & \alpha d_{2n} & \cdots & \alpha d_{3n-1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \alpha d_0 & \cdots & \alpha d_n & c_1 & \cdots & c_n & -\alpha d_0 & \cdots & -\alpha d_n & c_1 & \cdots & c_n \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \alpha d_{2n} & \cdots & \alpha d_{3n} & c_{2n+1} & \cdots & c_{3n} & -\alpha d_{2n} & \cdots & -\alpha d_{3n} & c_{2n+1} & \cdots & c_{3n} \end{vmatrix} \\ &= \begin{vmatrix} c_0 & \cdots & c_n & \alpha d_0 & \cdots & \alpha d_{n-1} \\ \vdots & & \vdots & \vdots & & \vdots \\ c_{2n} & \cdots & c_{3n} & \alpha d_{2n} & \cdots & \alpha d_{3n-1} \end{vmatrix} \begin{vmatrix} -\alpha d_0 & \cdots & -\alpha d_n & c_1 & \cdots & c_n \\ \vdots & & \vdots & \vdots & & \vdots \\ -\alpha d_{2n} & \cdots & -\alpha d_{3n} & c_{2n+1} & \cdots & c_{3n} \end{vmatrix} \\ &= |\alpha^{2n+1} P_n Q_n|, \end{aligned}$$

where we used a column operation in the second equality to cancel out the $c_i's$ in the upper third block. The proof of all remaining equalities proceeds essentially in the same

way. Let us show that $|H_{2n+1}|=|\alpha^{2n+2}H_nR_n|$. As for the previous relation, we have that

$$
\begin{aligned}
|H_{2n+1}| &= \begin{vmatrix} c_0 & c_1 & \cdots & c_{2n+1} & d_0 & d_1 & \cdots & d_{2n+1} \\ c_1 & c_2 & \cdots & c_{2n+2} & d_1 & d_2 & \cdots & d_{2n+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{4n+2} & c_{4n+3} & \cdots & c_{6n+3} & d_{4n+2} & d_{4n+3} & \cdots & d_{6n+3} \\ c_{4n+3} & c_{4n+4} & \cdots & c_{6n+4} & d_{4n+3} & d_{4n+4} & \cdots & d_{6n+4} \end{vmatrix} \\
&= \begin{vmatrix} c_0 & \cdots & c_{2n} & c_1 & \cdots & c_{2n+1} & d_0 & \cdots & d_{2n} & d_1 & \cdots & d_{2n+1} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{4n+2} & \cdots & c_{6n+2} & c_{4n+3} & \cdots & c_{6n+3} & d_{4n+2} & \cdots & d_{6n+2} & d_{4n+3} & \cdots & d_{6n+3} \\ c_1 & \cdots & c_{2n+1} & c_2 & \cdots & c_{2n+2} & d_1 & \cdots & d_{2n+1} & d_2 & \cdots & d_{2n+2} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{4n+3} & \cdots & c_{6n+3} & c_{4n+4} & \cdots & c_{6n+4} & d_{4n+3} & \cdots & d_{6n+3} & d_{4n+4} & \cdots & d_{6n+4} \end{vmatrix} \\
&= \begin{vmatrix} C^{(0)}_{2n+1,n} & \alpha D^{(0)}_{2n+1,n} & 0 & 0 \\ \alpha D^{(0)}_{2n+1,n} & C^{(1)}_{2n+2,n+1} & -\alpha D^{(0)}_{2n+1,n} & C^{(1)}_{2n+2,n+1} \end{vmatrix} \\
&= \begin{vmatrix} c_0 & \cdots & c_n & \alpha d_0 & \cdots & \alpha d_n \\ \vdots & & \vdots & \vdots & & \vdots \\ c_{2n+1} & \cdots & c_{3n+1} & \alpha d_{2n+1} & \cdots & \alpha d_{3n+1} \end{vmatrix} \begin{vmatrix} -\alpha d_0 & \cdots & -\alpha d_n & c_1 & \cdots & c_{n+1} \\ \vdots & & \vdots & \vdots & & \vdots \\ -\alpha d_{2n+1} & \cdots & -\alpha d_{3n+1} & c_{2n+2} & \cdots & c_{3n+2} \end{vmatrix} \\
&= |\alpha^{2n+2}H_nR_n|.
\end{aligned}
$$

Let us now consider the relations for P_{2n} and P_{2n+1}. We have that

$$
\begin{aligned}
|P_{2n}| &= \begin{vmatrix} c_0 & c_1 & \cdots & c_{2n} & d_0 & d_1 & \cdots & d_{2n-1} \\ c_1 & c_2 & \cdots & c_{2n+1} & d_1 & d_2 & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ c_{4n} & c_{4n+1} & \cdots & c_{6n} & d_{4n} & d_{4n+1} & \cdots & d_{6n-1} \end{vmatrix} \\
&= \begin{vmatrix} c_0 & \cdots & c_{2n} & c_1 & \cdots & c_{2n-1} & d_0 & \cdots & d_{2n-2} & d_1 & \cdots & d_{2n-1} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{4n} & \cdots & c_{6n} & c_{4n+1} & \cdots & c_{6n-1} & d_{4n} & \cdots & d_{6n-2} & d_{4n+1} & \cdots & d_{6n-1} \\ c_1 & \cdots & c_{2n+1} & c_2 & \cdots & c_{2n} & d_1 & \cdots & d_{2n-1} & d_2 & \cdots & d_{2n} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{4n-1} & \cdots & c_{6n-1} & c_{4n} & \cdots & c_{6n-2} & d_{4n-1} & \cdots & d_{6n-3} & d_{4n} & \cdots & d_{6n-2} \end{vmatrix} \\
&= \begin{vmatrix} C^{(0)}_{2n,n} & \alpha D^{(0)}_{2n,n-1} & C^{(0)}_{2n,n-1} & 0 \\ \alpha D^{(0)}_{2n-1,n} & C^{(1)}_{2n,n} & 0 & C^{(1)}_{2n,n} \end{vmatrix} \\
&= \begin{vmatrix} C^{(0)}_{2n,n} & \alpha D^{(0)}_{2n,n-1} & 0 & 0 \\ \alpha D^{(0)}_{2n-1,n} & C^{(1)}_{2n,n} & -\alpha D^{(0)}_{2n-1,n-1} & C^{(1)}_{2n,n} \end{vmatrix} \\
&= \alpha^{2n} \begin{vmatrix} C^{(0)}_{2n,n} & D^{(0)}_{2n,n-1} \end{vmatrix} \begin{vmatrix} D^{(0)}_{2n-1,n-1} & C^{(1)}_{2n,n} \end{vmatrix} \\
&= |\alpha^{2n}P_nR_{n-1}|
\end{aligned}
$$

and

$$
\begin{aligned}
|P_{2n+1}| &= \begin{vmatrix} c_0 & c_1 & \cdots & c_{2n+1} & d_0 & d_1 & \cdots & d_{2n} \\ c_1 & c_2 & \cdots & c_{2n+2} & d_1 & d_2 & \cdots & d_{2n+1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ c_{4n+2} & c_{4n+3} & \cdots & c_{6n+3} & d_{4n+2} & d_{4n+3} & \cdots & d_{6n+2} \end{vmatrix} \\
&= \begin{vmatrix} c_0 & \cdots & c_{2n} & c_1 & \cdots & c_{2n+1} & d_0 & \cdots & d_{2n} & d_1 & \cdots & d_{2n-1} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{4n+2} & \cdots & c_{6n+2} & c_{4n+3} & \cdots & c_{6n+3} & d_{4n+2} & \cdots & d_{6n+2} & d_{4n+3} & \cdots & d_{6n+1} \\ c_1 & \cdots & c_{2n+1} & c_2 & \cdots & c_{2n+2} & d_1 & \cdots & d_{2n+1} & d_2 & \cdots & d_{2n} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{4n+1} & \cdots & c_{6n+1} & c_{4n+2} & \cdots & c_{6n+2} & d_{4n+1} & \cdots & d_{6n+1} & d_{4n+2} & \cdots & d_{6n} \end{vmatrix} \\
&= \begin{vmatrix} C^{(0)}_{2n+1,n} & \alpha D^{(0)}_{2n+1,n} & C^{(0)}_{2n+1,n} & 0 \\ \alpha D^{(0)}_{2n,n} & C^{(1)}_{2n+1,n+1} & 0 & C^{(1)}_{2n+1,n} \end{vmatrix} \\
&= \begin{vmatrix} C^{(0)}_{2n+1,n} & \alpha D^{(0)}_{2n+1,n} & 0 & 0 \\ \alpha D^{(0)}_{2n,n} & C^{(1)}_{2n+1,n+1} & -\alpha D^{(0)}_{2n,n} & C^{(1)}_{2n+1,n} \end{vmatrix} \\
&= \alpha^{2n+2} \begin{vmatrix} C^{(0)}_{2n+1,n} & D^{(0)}_{2n+1,n} \end{vmatrix} \begin{vmatrix} D^{(0)}_{2n,n} & C^{(1)}_{2n+1,n} \end{vmatrix} \\
&= |\alpha^{2n+2} H_n Q_n|.
\end{aligned}
$$

Analogously, for Q_{2n} and Q_{2n+1} we have that

$$
\begin{aligned}
|Q_{2n}| &= \begin{vmatrix} c_1 & c_2 & \cdots & c_{2n} & d_0 & d_1 & \cdots & d_{2n} \\ c_2 & c_3 & \cdots & c_{2n+1} & d_1 & d_2 & \cdots & d_{2n+1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ c_{4n+1} & c_{4n+2} & \cdots & c_{6n} & d_{4n} & d_{4n+1} & \cdots & d_{6n} \end{vmatrix} \\
&= \begin{vmatrix} c_1 & \cdots & c_{2n-1} & c_2 & \cdots & c_{2n} & d_0 & \cdots & d_{2n} & d_1 & \cdots & d_{2n-1} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{4n+1} & \cdots & c_{6n-1} & c_{4n+2} & \cdots & c_{6n} & d_{4n} & \cdots & d_{6n} & d_{4n+1} & \cdots & d_{6n-1} \\ c_2 & \cdots & c_{2n} & c_3 & \cdots & c_{2n+1} & d_1 & \cdots & d_{2n+1} & d_2 & \cdots & d_{2n} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{4n} & \cdots & c_{6n-2} & c_{4n+1} & \cdots & c_{6n-1} & d_{4n-1} & \cdots & d_{6n-1} & d_{4n} & \cdots & d_{6n-2} \end{vmatrix} \\
&= \begin{vmatrix} \alpha D^{(0)}_{2n,n-1} & C^{(1)}_{2n+1,n} & C^{(0)}_{2n,n} & 0 \\ C^{(1)}_{2n,n} & \alpha D^{(1)}_{2n,n} & 0 & C^{(1)}_{2n,n} \end{vmatrix} \\
&= \begin{vmatrix} \alpha D^{(0)}_{2n,n-1} & C^{(1)}_{2n+1,n} & C^{(0)}_{2n,n} & 0 \\ 0 & \alpha D^{(1)}_{2n,n} & 0 & C^{(1)}_{2n,n} \end{vmatrix} \\
&= \begin{vmatrix} \alpha D^{(0)}_{2n,n-1} & C^{(0)}_{2n,n} & C^{(1)}_{2n+1,n} & 0 \\ 0 & 0 & \alpha D^{(1)}_{2n,n} & C^{(1)}_{2n,n} \end{vmatrix} \\
&= \alpha^{2n} \begin{vmatrix} D^{(0)}_{2n,n-1} & C^{(0)}_{2n,n} \end{vmatrix} \begin{vmatrix} D^{(1)}_{2n,n} & C^{(1)}_{2n,n} \end{vmatrix} = |\alpha^{2n} P_n T_n|
\end{aligned}
$$

and

$$
\begin{aligned}
|Q_{2n+1}| &= \begin{vmatrix} c_1 & c_2 & \cdots & c_{2n+1} & d_0 & d_1 & \cdots & d_{2n+1} \\ c_2 & c_3 & \cdots & c_{2n+2} & d_1 & d_2 & \cdots & d_{2n+2} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ c_{4n+3} & c_{4n+4} & \cdots & c_{6n+3} & d_{4n+2} & d_{4n+3} & \cdots & d_{6n+3} \end{vmatrix} \\
&= \begin{vmatrix} c_1 & \cdots & c_{2n+1} & c_2 & \cdots & c_{2n} & d_0 & \cdots & d_{2n} & d_1 & \cdots & d_{2n+1} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{4n+3} & \cdots & c_{6n+3} & c_{4n+4} & \cdots & c_{6n+2} & d_{4n+2} & \cdots & d_{6n+2} & d_{4n+3} & \cdots & d_{6n+3} \\ c_2 & \cdots & c_{2n+2} & c_3 & \cdots & c_{2n+1} & d_1 & \cdots & d_{2n+1} & d_2 & \cdots & d_{2n+2} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{4n+2} & \cdots & c_{6n+2} & c_{4n+3} & \cdots & c_{6n+1} & d_{4n+1} & \cdots & d_{6n+1} & d_{4n+2} & \cdots & d_{6n+2} \end{vmatrix} \\
&= \begin{vmatrix} \alpha D^{(0)}_{2n+1,n} & C^{(1)}_{2n+2,n} & C^{(0)}_{2n+1,n} & 0 \\ C^{(1)}_{2n+1,n+1} & \alpha D^{(1)}_{2n+1,n} & 0 & C^{(1)}_{2n+1,n+1} \end{vmatrix} \\
&= \begin{vmatrix} \alpha D^{(0)}_{2n+1,n} & C^{(1)}_{2n+2,n} & C^{(0)}_{2n+1,n} & 0 \\ 0 & \alpha D^{(1)}_{2n+1,n} & 0 & C^{(1)}_{2n+1,n+1} \end{vmatrix} \\
&= \begin{vmatrix} \alpha D^{(0)}_{2n+1,n} & C^{(0)}_{2n+1,n} & C^{(1)}_{2n+2,n} & 0 \\ 0 & 0 & \alpha D^{(1)}_{2n+1,n} & C^{(1)}_{2n+1,n+1} \end{vmatrix} \\
&= |\alpha^{2n+1}| \begin{vmatrix} D^{(0)}_{2n+1,n} & C^{(0)}_{2n+1,n} \end{vmatrix} \begin{vmatrix} D^{(1)}_{2n+1,n} & C^{(1)}_{2n+1,n+1} \end{vmatrix} = |\alpha^{2n+1} H_n U_n|.
\end{aligned}
$$

For R_{2n} and R_{2n+1} we have that

$$
\begin{aligned}
|R_{2n}| &= \begin{vmatrix} c_1 & c_2 & \cdots & c_{2n+1} & d_0 & d_1 & \cdots & d_{2n} \\ c_2 & c_3 & \cdots & c_{2n+2} & d_1 & d_2 & \cdots & d_{2n+1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ c_{4n+2} & c_{4n+3} & \cdots & c_{6n+2} & d_{4n+1} & d_{4n+2} & \cdots & d_{6n+1} \end{vmatrix} \\
&= \begin{vmatrix} c_1 & \cdots & c_{2n+1} & c_2 & \cdots & c_{2n} & d_0 & \cdots & d_{2n} & d_1 & \cdots & d_{2n-1} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{4n+1} & \cdots & c_{6n+1} & c_{4n+2} & \cdots & c_{6n} & d_{4n} & \cdots & d_{6n} & d_{4n+1} & \cdots & d_{6n-1} \\ c_2 & \cdots & c_{2n+2} & c_3 & \cdots & c_{2n+1} & d_1 & \cdots & d_{2n+1} & d_2 & \cdots & d_{2n} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{4n+2} & \cdots & c_{6n+2} & c_{4n+3} & \cdots & c_{6n+1} & d_{4n+1} & \cdots & d_{6n+1} & d_{4n+2} & \cdots & d_{6n} \end{vmatrix} \\
&= \begin{vmatrix} \alpha D^{(0)}_{2n,n} & C^{(1)}_{2n+1,n} & C^{(0)}_{2n,n} & 0 \\ C^{(1)}_{2n+1,n+1} & \alpha D^{(1)}_{2n+1,n} & 0 & C^{(1)}_{2n+1,n} \end{vmatrix} \\
&= \begin{vmatrix} \alpha D^{(0)}_{2n,n} & C^{(1)}_{2n+1,n} & C^{(0)}_{2n,n} & -\alpha D^{(0)}_{2n,n-1} \\ C^{(1)}_{2n+1,n+1} & \alpha D^{(1)}_{2n+1,n} & 0 & 0 \end{vmatrix} \\
&= \alpha^{2n} \begin{vmatrix} C^{(0)}_{2n,n} & D^{(0)}_{2n,n-1} \end{vmatrix} \begin{vmatrix} C^{(1)}_{2n+1,n+1} & D^{(1)}_{2n+1,n} \end{vmatrix} = |\alpha^{2n} P_n U_n|
\end{aligned}
$$

and

$$
\begin{aligned}
|R_{2n+1}| &= \begin{vmatrix} c_1 & c_2 & \cdots & c_{2n+2} & d_0 & d_1 & \cdots & d_{2n+1} \\ c_2 & c_3 & \cdots & c_{2n+3} & d_1 & d_2 & \cdots & d_{2n+2} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ c_{4n+4} & c_{4n+5} & \cdots & c_{6n+5} & d_{4n+3} & d_{4n+4} & \cdots & d_{6n+4} \end{vmatrix} \\
&= \begin{vmatrix} c_1 & \cdots & c_{2n+1} & c_2 & \cdots & c_{2n+2} & d_0 & \cdots & d_{2n} & d_1 & \cdots & d_{2n+1} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{4n+3} & \cdots & c_{6n+3} & c_{4n+4} & \cdots & c_{6n+4} & d_{4n+2} & \cdots & d_{6n+2} & d_{4n+3} & \cdots & d_{6n+3} \\ c_2 & \cdots & c_{2n+2} & c_3 & \cdots & c_{2n+3} & d_1 & \cdots & d_{2n+1} & d_2 & \cdots & d_{2n+2} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{4n+4} & \cdots & c_{6n+4} & c_{4n+5} & \cdots & c_{6n+5} & d_{4n+3} & \cdots & d_{6n+3} & d_{4n+4} & \cdots & d_{6n+4} \end{vmatrix} \\
&= \begin{vmatrix} \alpha D^{(0)}_{2n+1,n} & C^{(1)}_{2n+2,n+1} & C^{(0)}_{2n+1,n} & 0 \\ C^{(1)}_{2n+2,n+1} & \alpha D^{(1)}_{2n+2,n+1} & 0 & C^{(1)}_{2n+2,n+1} \end{vmatrix} \\
&= \begin{vmatrix} \alpha D^{(0)}_{2n+1,n} & C^{(1)}_{2n+2,n+1} & C^{(0)}_{2n+1,n} & 0 \\ 0 & \alpha D^{(1)}_{2n+2,n+1} & 0 & C^{(1)}_{2n+2,n+1} \end{vmatrix} \\
&= \begin{vmatrix} \alpha D^{(0)}_{2n+1,n} & C^{(0)}_{2n+1,n} & C^{(1)}_{2n+2,n+1} & 0 \\ 0 & 0 & \alpha D^{(1)}_{2n+2,n+1} & C^{(1)}_{2n+2,n+1} \end{vmatrix} \\
&= |\alpha^{2n+2}| \begin{vmatrix} D^{(0)}_{2n+1,n} & C^{(0)}_{2n+1,n} \end{vmatrix} \begin{vmatrix} D^{(1)}_{2n+2,n+1} & C^{(1)}_{2n+2,n+1} \end{vmatrix} \\
&= |\alpha^{2n+2} H_n T_{n+1}|.
\end{aligned}
$$

For T_{2n} and T_{2n+1} we have that

$$
\begin{aligned}
|T_{2n}| &= \begin{vmatrix} c_1 & c_2 & \cdots & c_{2n} & d_1 & d_2 & \cdots & d_{2n} \\ c_2 & c_3 & \cdots & c_{2n+1} & d_2 & d_3 & \cdots & d_{2n+1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ c_{4n} & c_{4n+1} & \cdots & c_{6n-1} & d_{4n} & d_{4n+1} & \cdots & d_{6n-1} \end{vmatrix} \\
&= \begin{vmatrix} c_1 & \cdots & c_{2n-1} & c_2 & \cdots & c_{2n} & d_1 & \cdots & d_{2n-1} & d_2 & \cdots & d_{2n} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{4n-1} & \cdots & c_{6n-3} & c_{4n} & \cdots & c_{6n-2} & d_{4n-1} & \cdots & d_{6n-3} & d_{4n} & \cdots & d_{6n-2} \\ c_2 & \cdots & c_{2n} & c_3 & \cdots & c_{2n+1} & d_2 & \cdots & d_{2n} & d_3 & \cdots & d_{2n+1} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{4n} & \cdots & c_{6n-2} & c_{4n+1} & \cdots & c_{6n-1} & d_{4n} & \cdots & d_{6n-2} & d_{4n+1} & \cdots & d_{6n-1} \end{vmatrix} \\
&= \begin{vmatrix} \alpha D^{(0)}_{2n-1,n-1} & C^{(1)}_{2n,n} & 0 & C^{(1)}_{2n,n} \\ C^{(1)}_{2n,n} & \alpha D^{(1)}_{2n,n} & C^{(1)}_{2n,n} & 0 \end{vmatrix} \\
&= \begin{vmatrix} \alpha D^{(0)}_{2n-1,n-1} & C^{(1)}_{2n,n} & 0 & 0 \\ C^{(1)}_{2n,n} & \alpha D^{(1)}_{2n,n} & C^{(1)}_{2n,n} & -\alpha D^{(1)}_{2n,n} \end{vmatrix} \\
&= |\alpha^{2n}| \begin{vmatrix} D^{(0)}_{2n-1,n-1} & C^{(1)}_{2n,n} \end{vmatrix} \begin{vmatrix} D^{(1)}_{2n,n} & C^{(1)}_{2n,n} \end{vmatrix} = |\alpha^{2n} R_{n-1} T_n|
\end{aligned}
$$

and

$$
\begin{aligned}
|T_{2n+1}| &= \begin{vmatrix} c_1 & c_2 & \cdots & c_{2n+1} & d_1 & d_2 & \cdots & d_{2n+1} \\ c_2 & c_3 & \cdots & c_{2n+2} & d_2 & d_3 & \cdots & d_{2n+2} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ c_{4n+2} & c_{4n+3} & \cdots & c_{6n+2} & d_{4n+2} & d_{4n+3} & \cdots & d_{6n+2} \end{vmatrix} \\
&= \begin{vmatrix} c_1 & \cdots & c_{2n+1} & c_2 & \cdots & c_{2n} & d_1 & \cdots & d_{2n+1} & d_2 & \cdots & d_{2n} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{4n+1} & \cdots & c_{6n+1} & c_{4n+2} & \cdots & c_{6n} & d_{4n+1} & \cdots & d_{6n+1} & d_{4n+2} & \cdots & d_{6n} \\ c_2 & \cdots & c_{2n+2} & c_3 & \cdots & c_{2n+1} & d_2 & \cdots & d_{2n+2} & d_3 & \cdots & d_{2n+1} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{4n+2} & \cdots & c_{6n+2} & c_{4n+3} & \cdots & c_{6n+1} & d_{4n+2} & \cdots & d_{6n+2} & d_{4n+3} & \cdots & d_{6n+1} \end{vmatrix} \\
&= \begin{vmatrix} \alpha D^{(0)}_{2n,n} & C^{(1)}_{2n+1,n} & 0 & C^{(1)}_{2n+1,n} \\ C^{(1)}_{2n+1,n+1} & \alpha D^{(1)}_{2n+1,n} & C^{(1)}_{2n+1,n+1} & 0 \end{vmatrix} \\
&= \begin{vmatrix} \alpha D^{(0)}_{2n,n} & C^{(1)}_{2n+1,n} & 0 & 0 \\ C^{(1)}_{2n+1,n+1} & \alpha D^{(1)}_{2n+1,n} & C^{(1)}_{2n+1,n+1} & -\alpha D^{(1)}_{2n+1,n} \end{vmatrix} \\
&= |\alpha^{2n+1}| \begin{vmatrix} D^{(0)}_{2n,n} & C^{(1)}_{2n+1,n} \end{vmatrix} \begin{vmatrix} C^{(1)}_{2n+1,n+1} & D^{(1)}_{2n+1,n} \end{vmatrix} \\
&= |\alpha^{2n+1} Q_n U_n|.
\end{aligned}
$$

Finally, for U_{2n} and U_{2n+1} we have that

$$
\begin{aligned}
|U_{2n}| &= \begin{vmatrix} c_1 & c_2 & \cdots & c_{2n+1} & d_1 & d_2 & \cdots & d_{2n} \\ c_2 & c_3 & \cdots & c_{2n+2} & d_2 & d_3 & \cdots & d_{2n+1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ c_{4n+1} & c_{4n+2} & \cdots & c_{6n+1} & d_{4n+1} & d_{4n+2} & \cdots & d_{6n} \end{vmatrix} \\
&= \begin{vmatrix} c_1 & \cdots & c_{2n+1} & c_2 & \cdots & c_{2n} & d_1 & \cdots & d_{2n-1} & d_2 & \cdots & d_{2n} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{4n+1} & \cdots & c_{6n+1} & c_{4n+2} & \cdots & c_{6n} & d_{4n+1} & \cdots & d_{6n-1} & d_{4n+2} & \cdots & d_{6n} \\ c_2 & \cdots & c_{2n+2} & c_3 & \cdots & c_{2n+1} & d_2 & \cdots & d_{2n} & d_3 & \cdots & d_{2n+1} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{4n} & \cdots & c_{6n} & c_{4n+1} & \cdots & c_{6n-1} & d_{4n} & \cdots & d_{6n-2} & d_{4n+1} & \cdots & d_{6n-1} \end{vmatrix} \\
&= \begin{vmatrix} \alpha D^{(0)}_{2n,n} & C^{(1)}_{2n+1,n} & 0 & C^{(1)}_{2n+1,n} \\ C^{(1)}_{2n,n+1} & \alpha D^{(1)}_{2n,n} & C^{(1)}_{2n,n} & 0 \end{vmatrix} \\
&= \begin{vmatrix} \alpha D^{(0)}_{2n,n} & C^{(1)}_{2n+1,n} & 0 & 0 \\ C^{(1)}_{2n,n+1} & \alpha D^{(1)}_{2n,n} & C^{(1)}_{2n,n} & -\alpha D^{(1)}_{2n,n} \end{vmatrix} \\
&= |\alpha^{2n+1}| \begin{vmatrix} D^{(0)}_{2n,n} & C^{(1)}_{2n+1,n} \end{vmatrix} \begin{vmatrix} C^{(1)}_{2n,n} & D^{(1)}_{2n,n} \end{vmatrix} \\
&= |\alpha^{2n+1} Q_n T_n|
\end{aligned}
$$

and

$$
\begin{aligned}
|U_{2n+1}| &= \begin{vmatrix} c_1 & c_2 & \cdots & c_{2n+2} & d_1 & d_2 & \cdots & d_{2n+1} \\ c_2 & c_3 & \cdots & c_{2n+3} & d_2 & d_3 & \cdots & d_{2n+2} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ c_{4n+3} & c_{4n+4} & \cdots & c_{6n+4} & d_{4n+3} & d_{4n+4} & \cdots & d_{6n+3} \end{vmatrix} \\
&= \begin{vmatrix} c_1 & \cdots & c_{2n+1} & c_2 & \cdots & c_{2n+2} & d_1 & \cdots & d_{2n+1} & d_2 & \cdots & d_{2n} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{4n+3} & \cdots & c_{6n+3} & c_{4n+4} & \cdots & c_{6n+4} & d_{4n+3} & \cdots & d_{6n+3} & d_{4n+4} & \cdots & d_{6n+2} \\ c_2 & \cdots & c_{2n+2} & c_3 & \cdots & c_{2n+3} & d_2 & \cdots & d_{2n+2} & d_3 & \cdots & d_{2n+1} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{4n+2} & \cdots & c_{6n+2} & c_{4n+3} & \cdots & c_{6n+3} & d_{4n+2} & \cdots & d_{6n+2} & d_{4n+3} & \cdots & d_{6n+1} \end{vmatrix} \\
&= \begin{vmatrix} \alpha D^{(0)}_{2n+1,n} & C^{(1)}_{2n+2,n+1} & 0 & C^{(1)}_{2n+2,n} \\ C^{(1)}_{2n+1,n+1} & \alpha D^{(1)}_{2n+1,n+1} & C^{(1)}_{2n+1,n+1} & 0 \end{vmatrix} \\
&= \begin{vmatrix} \alpha D^{(0)}_{2n+1,n} & C^{(1)}_{2n+2,n+1} & 0 & 0 \\ C^{(1)}_{2n+1,n+1} & \alpha D^{(1)}_{2n+1,n+1} & C^{(1)}_{2n+1,n+1} & -\alpha D^{(1)}_{2n+1,n} \end{vmatrix} \\
&= |\alpha^{2n+1}| \begin{vmatrix} D^{(0)}_{2n+1,n} & C^{(1)}_{2n+2,n+1} \end{vmatrix} \begin{vmatrix} C^{(1)}_{2n+1,n+1} & D^{(1)}_{2n+1,n} \end{vmatrix} \\
&= |\alpha^{2n+1} R_n U_n|.
\end{aligned}
$$

□

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Department of Mathematics, North Dakota State University, Fargo ND, USA

Email address: reynold.fregoli@ndsu.edu

Department of Mathematics, University of Michigan, Ann Arbor MI, USA

Email address: uyizhang@umich.edu